A VARIANCE ANALYSIS OF THE PHD AND k-PHD SINGLE-TONE FREQUENCY ESTIMATORS: RANDOM PHASE CASE

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This thesis titled "A VARIANCE ANALYSIS OF THE PHD AND k-PHD SINGLE TONE FREQUENCY ESTIMATORS: RANDOM PHASE CASE" and prepared by Keyta BETOFE MBOYO has been accepted as a **MSc THESIS** in Bursa Uludağ University Graduate School of Natural and Applied Sciences, Department of Electronics Engineering following a unanimous vote of the jury below.

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ÖZET

Yüksek Lisans

PHD ve k-PHD TEK-TON FREKANS KESTİRİCİLERİNİN BİR DEĞİŞİNTİ ANALİZİ: RASGELE FAZ DURUMU

Bursa Uludağ Üniversitesi

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İyi bilinen Pisarenko harmonik ayrışım (PHD) metodunun bir genelleştirilmesi olan k-PHD metodu gürültü içindeki bir reel rasgele fazlı sinüsün frekans kestirimi için ele alınmıştır. Basit bir değişinti analiz tekniği kullanılarak, k-PHD frekans değişintisinin bir kesin ifadesi çıkarılmıştır. Yeteri kadar geniş veri uzunlukları ve yüksek işaret gürültü oranları için bir yaklaşık k-PHD değişinti formülü de verilmektedir. Kuramsal sonuçları teyit eden bilgisayar benzetimleri dâhil edilmiştir.

Anahtar Kelimeler: Frekans kestirimi, Reel sinus, PHD metodu, k-PHD metodu, Değişinti analizi.
2019, viii + 46 sayfa.

ABSTRACT

MSc Thesis

A VARIANCE ANALYSIS OF THE PHD AND k-PHD SINGLE TONE FREQUENCY ESTIMATORS: RANDOM PHASE CASE

Bursa Uludağ University
Graduate School of Natural and Applied Sciences
Department of Electronics Engineering

Supervisor: Prof. Dr. Erdoğan DİLAVEROĞLU

The well-known Pisarenko harmonic decomposition (PHD) method and the k-PHD method, which is a generalization of the PHD method, are considered for frequency estimation of a single real random-phased sinusoid in noise. With the use of a simple variance analysis technique, exact expressions of the PHD and k-PHD variances of frequency are derived. Approximate PHD and k-PHD variance formulas which are valid for sufficiently large data lengths and high signal-to-noise ratios are also given. Computer simulations are included to corroborate the theoretical developments.

Key words: Frequency estimation, Real sinusoid, PHD method, k-PHD method, Variance analysis.
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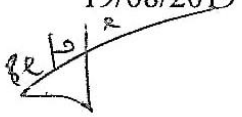
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CONTENTS

Page
ÖZET i
ABSTRACT ii
ACKNOWLEDGEMENT iii
SYMBOL AND ABBREVIATIONS INDEX iv
FIGURES v
TABLES……. vi
1. INTRODUCTION 1
1.1. Frequency Estimation 1
1.2. Estimation Theory 3
2. THEORETICAL FUNDAMENTALS 6
2.1. Pisarenko's harmonic decomposition approach 6
2.2. Pisarenko's method for frequency estimation of real tone 9
2.3. k-PHD method for frequency estimation of real tone 10
2.4. Description of the method 11
3. MATERIAL AND METHODS 14
3.1. Variance analysis of PHD 14
3.1.1. Data model and PHD method 14
3.1.2. Exact PHD Variance development 15
3.2. Variance analysis of k-PHD 18
3.2.1. Data model and k-PHD method 18
3.2.2. Exact k-PHD Variance development 19
4. RESULTS AND DISCUSSION 36
4.1. Numerical examples for PHD 36
4.2. Numerical examples for k-PHD 40
5. CONCLUSION 44
REFERENCES 45
RESUME 46

SYMBOL AND ABBREVIATIONS INDEX

Symbol	description
ω	Frequency of the Sinusoid, Unknown Parameter
$\widehat{\omega}$	Estimate frequency
N	Number of Samples
α	Amplitude, unknown constant
ϕ	Phase, random variable
var	Variance
σ^2	Variance of Noise
E{·}	Expected value operator

Abbreviations index	description
SNR	Signal-to-noise ratio
MSE	Mean Squared Error
CRLB	Cramer-Rao Lower Bound
AWGN	Additive White Gaussian Noise
PHD	Pisarenko Harmonic Decomposition

FIGURES

Page

Figure 4.1.Variances of Frequency versus ω at SNR=20 and N=8 36
Figure 4.2. Variances of Frequency versus ω at SNR=20 and N=32 37
Figure 4.3. Variances of Frequency versus ω at SNR=20 and N=12 37
Figure 4.4. Variances of Frequency versus SNR at ω =3π/8 and N=8 38
Figure 4.5. Variances of Frequency versus SNR at ω =3π/8 and N=32 39
Figure 4.6. Variances of Frequency versus SNR at ω =3π/8 and N=128 39
Figure 4.7. Variances of Frequency versus ω at SNR=20 dB and N=32. 40
Figure 4.8. Variances of Frequency versus ω at SNR=20 dB and N=128.....41
Figure 4.9. Variances of Frequency versus SNR at ω=3π/8 and N=128. 42

TABLES

Pages

Table 4.1. Frequency standard deviations versus N at SNR = 20 dB and w = 3π/8...43

1. INTRODUCTION

1.1 FREQUENCY ESTIMATION

In 1973, Vladilen Pisarenko published his fundamental work on frequency estimation for single-frequency sinusoids, entitled "harmonic decomposition of Pisarenko (PHD)". Since then, frequency estimation has become a topic of choice for statistical research processing signals using many PHD-based methods, such as reformulated PHD, modified PHD, modified covariance, and covariance modified correlation methods.
This research has also been extended to include estimating the frequency of several harmonic frequency signals.

One of the multiple frequency estimation methods is the multiple signal classification. The frequency estimate is located at the intersection of signal processing and statistical estimation theory. This type of estimation theory is known as parametric estimation. The parameters are unknown but deterministic quantities. A dataset depends typically on an unknown parameter and the aim is to determine the parameter based on the data or to define an estimator. And in this research, the data comes from a sinusoidal signal with noisy discrete time and the purpose is to determine its frequency.

İn general there are two main categories of frequency estimators: optimal estimators and non-optimal estimators. The optimal estimators are the most accurate estimators and the highest computational costs. One of the examples of an optimal estimator is Maximum likelihood frequency estimator. And compare to the Optimal Estimators, non-optimal estimators are less accurate, but their computational cost is less. The maximum likelihood estimation technique although it produces the most accurate estimator, it is rarely used in practice because the computational workload is so high. In fact, it cannot be used in applications requiring real-time measurements.

The signals of this domain of research are composed of a deterministic sinusoidal component and a random noise component. The random noise component usually comes from communication channel and can be modeled as Additive White Gaussian

Noise (AWGN). Determining the frequency is difficult due to the noise corruption, especially when the sinusoid amplitude is small compared to the noise variance. The ratio of the signal strength to the noise power is called the signal-to-noise ratio (SNR). When the SNR is small Frequency estimators generally do not perform well.

İn many applications Sinusoidal signals are widely. Among applications that use sinusoidal signals, the frequency must often be measured to determine the useful system information. For some applications, it may seem that a digital frequency counter is an easy way to determine the frequency. However, they have serious limitations. The Precision and resolution depend on the time base, usually a circuit using a crystal oscillator, sensitive to disturbances. The resolution is also proportional to the frequency of the oscillator of the time base and the preset time on which the frequency is measured. Jitter will rapidly degrade the accuracy of the measurement. Since applications such as radar often suffer from cross-talk, multipath, and intersymbol interference which can degrade signal quality and introduce jitter, a frequency counter should not be used.

The applications of Frequency estimation can be found in radar, sonar, communications, control theory, and more. It is commonly used in communications to determine the frequency of a carrier wave so the signal can be demodulated.

To find the distance of an object the time of flight can be estimated by performing frequency estimation in the frequency domain and then multiplied by the speed of light. Frequency estimation is also useful for signals whose frequencies appear shifted due to the Doppler Effect.

For an accurate and computationally efficient frequency estimator there is a need that performs well over a wide range of SNRs, frequencies, and number of samples. We need to have an accurate and precise estimator, what this means will be defined in the estimation theory section.

1.2 ESTIMATION THEORY

Estimation theory is a field of statistics whose objective is the estimation the values of population parameters from stochastic component data samples. An estimator is a function of samples used to estimate the value of a parameter often expressed as a formula. The value obtained from an estimator evaluated at the sample observed values is called an estimate. As types of parameter estimation we have the point estimate and the interval estimate. The point estimate uses sample data to compute a unique value that most closely resembles the true value of an unknown population. The interval estimate calculates a range of values instead of a single value. The frequency estimate falls into the point estimate category.

To evaluate the performance of an estimator, a bias, and a mean squared error (MSE), two critical measures are used. One could make an argument that computational complexity should be included in this list since there is a trade-off between estimator MSE and computational complexity. Because an estimator is a random variable the performance can be described by its probability density (PDF). Estimator precision is determined by its MSE and estimator accuracy is determined by its bias

Let consider θ as a parameter and $\hat{\theta}$ be a point estimator for θ. The Bias is defined as the difference between the expected value of the estimator and the true value of the parameter.

$$B(\hat{\theta}) = E\{\hat{\theta}\} - \theta \tag{1.1}$$

Where $B(\hat{\theta})$ is the bias operator and $E\{\cdot\}$ is the expected value operator. And we know that expected value is a weighted average.$\hat{\theta}$ is an unbiased estimator if $B(\hat{\theta})=0$. Otherwise, it is a biased estimator. An estimator is most accurate when it is unbiased; however, finding an unbiased estimator is not always possible. When this is the case finding an asymptotically unbiased estimator becomes the next best option. An asymptotically unbiased estimator is defined as an estimator whose bias goes to zero as the number of samples tends to infinity.

$$\lim_{N\to\infty} B(\hat{\theta}) \to 0$$

Many sub-optimal estimators are considered as the asymptotically unbiased. Assuming that an estimator is at least asymptotically unbiased, there are also some criteria that the variance and / or the MSE must meet for an estimator to be considered good. First of all, the difference between MSE and variance is that MSE is a function of bias and variance.

$$\mathrm{MSE}(\hat{\theta}) = E\{(\hat{\theta} - \theta)^2\} = \left[B(\hat{\theta})\right]^2 + var(\hat{\theta}) \tag{1.2}$$

where $var(\cdot)$ is the variance operator. A direct conclusion of this equation is that MSE and variance are equivalent when an estimator is unbiased. For the ideal case, the variance of an estimator will also go to zero as the number of samples tends to infinity. A consistent estimator is the one with both the bias and the variance go to zero as the number of samples tends to infinity. If the estimator variance meets the theoretical lower bound on the variance of an unbiased estimator, the estimator is known as efficient. This is one of the most important performance measures.

Cramer-Rao Lower Bound (CRLB) is the lower bound on the variance of an unbiased estimator. Like unbiased estimators, efficient estimators are rare and therefore often look for asymptotically efficient estimators. An unbiased estimator can not have a variance smaller than the CRLB.

$$var(\hat{\theta}) \geq \frac{1}{I(\theta)}$$

where the Fisher Information I(θ) is defined as

$$I(\theta) = E\left(\frac{\partial ln[L(\theta|x)]}{\partial \theta^2}\right)^2 = -E\left\{\frac{\partial^2 ln[L(\theta|x)]}{\partial \theta^2}\right\} \tag{1.3}$$

and the likelihood function $L(\theta \mid x)$ is defined as

$$L(\theta|x) = \prod_{n=1}^{N} p\,(x_n|\theta) \tag{1.4}$$

where $p(x_n \mid \theta)$ is a probability density function, therefore $L(\theta \mid x)$ is the joint density function for all observations. Note that $p(x_n \mid \theta)$ represents a family of distributions given parameter θ and not a conditional probability since θ is not a random variable. An estimator is optimal if it attains the CRLB. An estimator is considered to be good if at minimum it is asymptotically unbiased and gets close to meeting the CRLB.

There are many methods to find estimators. These methods include, but are not limited to, maximum likelihood estimation, method of moments (MOME), best linear unbiased estimation (BLUE), and maximum a posteriori estimation (MAP). For each method there are advantages and disadvantages. For instance, a ML estimator is optimal, but may be too difficult to calculate and/or implement. To use many of the estimation methods the data must be independent and identically distributed meaning each random variable has the same PDF and are mutually independent.

2. THEORETICAL FUNDAMENTALS

2.1 PISARENKO'S HARMONIC DECOMPOSITION APPROACH

The performance of the estimator may vary in terms of computational complexity, bias, and variance. Ideally, there would be an effective frequency estimator with a low compute workload. The discovery of such a frequency estimator has been the subject of research in recent decades and would be a major step forward if found. Many suboptimal frequency estimation methods, including some of the methods to be discussed in this chapter, use the linear prediction property of a sinusoid. This property makes it possible to predict the next sample of a sinusoid from the two previous samples, since the frequency is known. By exploiting this property, it is possible to build frequency estimators.

Pisarenko Harmonic Decomposition is a noise subspace decomposition method. Pisarenko showed that the frequency of a signal can be calculated from the eigenvector corresponding to the minimum eigenvalue of the autocorrelation matrix. The corresponding eigenvector is referred to as the minimum eigenvector. There are multiple ways to implement the algorithm. The algorithm proposed by Pisarenko will be presented first and the simplified version will be presented afterwards.

PHD assumes that the signal $x(n)$ is a sum of $p \geq 1$complex sinusoids in AWGN and that p is known. This method can be used for a real signal and will be formulated as such.

Using $x(n)$ the $(p+1) \times (p+1)$autocorrelation matrix Rx is estimated.
The general autocorrelation matrix is a Hermitian matrix and has the form

$$\mathrm{R_X} = P\begin{bmatrix} \mathrm{R_{XX}}(0) & \mathrm{R^*_{XX}}(1) & \mathrm{R^*_{XX}}(2) & \dots & \mathrm{R^*_{XX}}(N-1) \\ \mathrm{R_{XX}}(1) & \mathrm{R_{XX}}(0) & \mathrm{R^*_{XX}}(1) & \dots & \mathrm{R^*_{XX}}(N-2) \\ \mathrm{R_{XX}}(2) & \mathrm{R_{XX}}(1) & \mathrm{R_{XX}}(0) & \dots & \mathrm{R^*_{XX}}(N-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{R_{XX}}(N-1) & \mathrm{R_{XX}}(N-2) & \mathrm{R_{XX}}(N-3) & \dots & \mathrm{R_{XX}}(0) \end{bmatrix}$$

Where H is the conjugate transpose operator (Hermitian) and Rxx(k) is the sample AC from. The autocorrelation matrix of x(n)is a superposition of the AC matrices from the signal and the noise, i.e. $R_x = R_s + R_q$. For a real sinusoid with frequency=0 the signal AC matrix Rs is defined as

$$R_s = P\begin{bmatrix} 1 & \cos(\omega_0) & \cos(2\omega_0) & \dots & \cos((N-1)\,\omega_0) \\ \cos(\omega_0) & 1 & \cos(\omega_0) & \dots & \cos((N-2)\,\omega_0) \\ \cos(2\omega_0) & \cos(\omega_0) & 1 & \dots & \cos((N-3)\,\omega_0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos((N-1)\,\omega_0) & \cos((N-2)\,\omega_0) & \cos((N-3)\,\omega_0) & \dots & 1 \end{bmatrix}$$

Where $P = \frac{\alpha^2}{2}$ is the signal power. The rank of this matrix is one. The noise autocorrelation matrix is diagonal since noise is uncorrelated except at lag zero. Therefore $R_q = \sigma^2 \mathrm{I}$ where I is the identity matrix. The dimension of the noise subspace is one. Using the matrix properties of eigenvectors it is obvious that

$$R_x \mathrm{V_i} = (R_S + \sigma^2 \mathrm{I})\mathrm{V_i} = \lambda_s \mathrm{V_i} + \sigma^2 \mathrm{V_i} = (\lambda_s + \sigma^2)\mathrm{V_i} \qquad (2.1)$$

where λ_s is the eigenvalue of R_S and $\mathrm{V_i}$ are the eigenvectors of and R_x. If a vector

$$e = [1, \cos(\omega_0), \cos(2\omega_0), \dots, \cos((N-1)\omega_0)]^T \qquad (2.2)$$

is defined, R_S can be rewritten as $R_S = Pee^H$. R_S has one nonzero eigenvalue since its rank is one. Since

$$e = P(ee^H)e = Pe(e^H e) = NPe \qquad (2.3)$$

The nonzero eigenvalue is equal to NP. The vector e is also the only eigenvector of R_S. Therefore, the maximum eigenvalue is $\lambda_{max} = NP + \sigma^2$ and the minimum eigenvalue

is $\lambda_{min} = \sigma^2$. Since the noise eigenvector corresponds to λ_{min} it is also the minimum eigenvector $\mathrm{v}_{\min}$. This leads to the conclusion that

$$e^H \mathrm{v}_{\min} = \sum_{K=0}^{P} v_{min}(k) e^{-jk\omega_0} = 0 \tag{2.4}$$

Since eigenvectors of full rank Hermitian matrices are orthogonal. This has the same form as the Disrete Fourier Transform of the elements of $\mathrm{V}_{\min}$.Therefore, the Discrete Fourier Transform of the elements of $\mathrm{V}_{\min}$ is equal to zero at ω_0.

$$\mathrm{v}_{\min}\left(e^{-jk\omega_0}\right) = \sum_{K=0}^{p} v_{min}(k) e^{-jk\omega_0} = 0 \tag{2.5}$$

Recall that $p = 1$in the single frequency case. Similarly for the Z-transform,

$$\mathrm{v}_{\min}(z) = \sum_{K=0}^{p} v_{min}(k) z^{-k} = \left(1 - e^{j\omega_0} z^{-1}\right) \tag{2.6}$$

Solving for the root of this equation to find the frequency estimate results in

$$\widehat{\omega}_0 = cos^{-1}\left(Re\left\{e^{j\omega_0}\right\}\right) \tag{2.7}$$

Where $Re\{\cdot\}$ is an operator that only retains the real part of a complex number. This method is easily summarized as seen below.

Algorithm 1: Pisarenko Harmonic Decomposition

Input: signal $x(n)$

1. Calculate the $(p+1) \times (p+1)$ Autocorrelation Matrix $R_x = E\{xx^H\}$
2. Find eigenvalues and eigenvectors of R_x
3. Determine minimum eigenvector from the minimum eigenvalue
4. Compute Z-transform of $\mathrm{v}_{\min}$

5. Solve for the root of the resulting equation $\hat{\omega}_0 = cos^{-1}\left(Re\{e^{j\omega_0}\}\right)$

PHD is limited since it is sensitive to noise and is a biased estimator. As shown in bellow the variance goes to zero as the number of samples goes to infinity.

$$var(\hat{\omega}_0) = \frac{cos^2(2\omega_0) + cos^2(\omega_0)}{4\,N\gamma^2(2 + cos(2\omega_0))^2 + sin^2(\omega_0)} \tag{2.8}$$

Recall that γ is the SNR. Because the variance of the PHD estimator is proportional to N^{-1} and the CRLB is proportional to N^{-3} the PHD estimator can not attain the CRLB. Since PHD was introduced, several methods have refined it to reduce the bias. Unfortunately the variances of these other methods have not been able to meet the CRLB either. In later sections a couple of these methods are explored and contrasted with PHD.

2.2 PISARENKO'S METHOD FOR FREQUENCY ESTIMATION OF REAL TONE

The problem of estimating the frequency of a single tone in noise has been frequently studied in the signal processing literature due to its wide range of applications. Among the various estimation techniques, the so-called Pisarenko harmonic decomposition (PHD) method has been attracted people because of its ease of implementation. Several statistical analyses of the PHD method for frequency estimation of a single real sinusoid in noise have been carried out in the literature (see, e.g, [3.1],[3.2], and the references therein). A number of simple approximate expressions for the variance of the PHD frequency estimator have been derived (see, e.g, [3.1]); but these expressions are not valid for small number of data points and/or high signal-to-noise ratios (SNRs). More recently, an exact expression of the PHD frequency variance, which holds for moderate data lengths and/or SNRs, has been given in [3.2]. however all of the above-mentioned analyses consider the phase of the sinusoid as a constant parameter which does not change from one snapshot to another. We have assumed that the phase of the sinusoid is

a random variable which is uniformly distributed in an interval of length 2π. By utilizing the variance analysis technique employed in [3.2], but taking into accounts that phase is now a random variable, we derive an exact closed-form expression for the frequency variance of the PHD method. An approximate variance formula that holds for sufficiently data lengths and SNRs is also given.

2.3 k-PHD METHOD FOR FREQUENCY ESTIMATION OF REAL TONE

The problem of estimating the frequency of a single tone in noise has been frequently studied in the signal processing literature due to its wide range of applications. Among the various estimation techniques, the Pisarenko harmonic decomposition (PHD) method has attracted people because of its ease of implementation. Several statistical analyses of the PHD method have been carried out in the literature (see, e.g.., Sakai (1984), Chan and So (2003), and the references therein). By utilizing the variance analysis technique employed in Chan and So (2003), but taking into account that the phase is now a random variable, we derive an exact closed-form expression for the frequency variance of the PHD method. An approximate variance formula that holds for sufficiently large data lengths and SNRs is also given. Despite the fact that the PHD estimator can be implemented in a very simple way, its variance is, in general, much larger than Cramér-Rao lower bound (see, e.g., Chan and So, 2003).

İn order to reduce the variance of the PHD estimator, a two-step procedure called k-PHD method has been proposed in De Sabata et. al. (2007). The k-PHD method consists of estimating an integer multiple (k-multiple) of the frequency and using the PHD method in resolving the induced aliasing. For k=1 there is no ambiguity in the frequency estimates and the k-PHD reduces to the PHD. A variance expression of the k-PHD frequency estimator has also been given in De Sabata et al. (2007).

However, in the above-mentioned analysis of the PHD and k-PHD methods the phase of the sinusoid is considered as a constant parameter which does not change from one realization to another. İn this thesis, it is assumed that the phase of the sinusoid is a random variable which is uniformly distributed in an interval of length 2π. By utilizing

the variance analysis technique employed in Chan and So (2003) and De Sabata et al. (2007), and taking into account that the phase is now random variable, we derived an exact closed-form expression for the frequency variance of the k-PHD method. An approximate variance formula that holds for sufficiently large data lengths and SNRs is also given.

The variance expressions derived in our work can easily be adapted to the PHD estimator for the random-phased sinusoid case. The results for the PHD (means k=1) method have been presented, recently, in Keyta and Dilaveroğlu (2019).

2.4 DESCRIPTION OF THE METHOD

We consider the following discrete-time signal model

$$x(n) = s(n) + e(n) = \alpha \cos(\omega n + \phi) + e(n), \qquad n = 0,1, \dots, N-1 \quad (2.9)$$

Where the amplitude $\alpha > 0$, (angular) frequency $\omega\epsilon(0,\pi)$ and phase ϕ are deterministic but unknown quantities, and $e(n)$ is a Gaussian, white noise, uncorrelated with the signal, with zero-mean and variance σ^2. The signal-to-noise ratio is $SNR = \alpha^2/(2\sigma^2)$. Following (2.9), we review here briefly the derivation of the PHD estimation of a multiple of ω. The sinusoid obeys the following k'th order linear prediction equation for any fixed integer k:

$$s(n) - 2\cos(k\omega)\, s(n-k) + s(n-k) = 0 \quad (2.10)$$

(n denotes the discrete time). İt is possible to derive from (2.9) and (2.10) the following vector eqution:

$$\mathrm{x}_n^T \mathrm{a} = \mathrm{e}_n^T \mathrm{a} \quad (2.11)$$

Where

$\mathrm{x}_n = [x(n)\ \ x(n-k)\ \ x(n-2k)]^T$,

$\mathrm{e}_n = [e(n)\ \ e(n-k)\ \ e(n-2k)]^T$,

and $\mathrm{a} = [1 \ -2\cos(k\omega) \ 1)]^T$. İf both sides of (2.11) are premultiplied by x_n, the expected value is taken, and the fact that s and e are uncorrelated is taken into account, the following eigenequation results.

$$\mathrm{R_{XX}a} = \sigma^2\mathrm{a} \tag{2.12}$$

İn (2.12), $\mathrm{R_{XX}}$ is the signal autocorrelation matrix, which has the shape

$$\mathrm{R_{XX}} = \begin{bmatrix} r_0 & r_k & r_{2k} \\ r_k & r_0 & r_k \\ r_{2k} & r_k & r_0 \end{bmatrix} \tag{2.13}$$

$$r_k = E[x(n)x(n+k)]$$
$$= E[s(n)+\omega(n)][s(n+k)+\omega(n+k)]$$
$$= E[s(n)s(n+k)] + E[\omega(n)\omega(n+k)]$$
$$s(n)s(n+k) = \alpha^2\cos(\omega n+\phi)\cos(\omega(n+k)+\phi)$$
$$= \tfrac{\alpha^2}{2}[\cos(\omega(2n+k)+2\phi) + \cos(\omega k)]$$
$$E[s(n)s(n+k)] = \frac{\alpha^2}{2}\cos(\omega k)$$
$$r_k = \frac{\alpha^2}{2}\cos(\omega k) + \sigma^2\delta[k]$$
$$r_0 = \frac{\alpha^2}{2} + \sigma^2$$
$$r_{2k} = \frac{\alpha^2}{2}\cos(\omega 2k)$$
$$\begin{bmatrix} r_0 & r_k & r_{2k} \\ r_k & r_0 & r_k \\ r_{2k} & r_k & r_0 \end{bmatrix} = \frac{\alpha^2}{2}\begin{bmatrix} 1 & \cos(\omega k) & \cos(\omega 2k) \\ \cos(\omega k) & 1 & \cos(\omega k) \\ \cos(\omega 2k) & \cos(\omega k) & 1 \end{bmatrix} + \sigma^2\mathrm{I} = \mathrm{A} + \sigma^2\mathrm{I}$$

$(\mathrm{A}+\sigma^2\mathrm{I})\mathrm{a} = \sigma^2\mathrm{a}$ becomes $\mathrm{Aa} = 0$

$$\begin{bmatrix} 1 & \cos(\omega k) & \cos(\omega 2k) \\ \cos(\omega k) & 1 & \cos(\omega k) \\ \cos(\omega 2k) & \cos(\omega k) & 1 \end{bmatrix} \rightarrow$$

$$\begin{matrix} s_2 - \cos(\omega k)\, s_1 \\ s_3 - \cos(\omega 2k)\, s_1 \end{matrix} \begin{bmatrix} 1 & \cos(\omega k) & \cos(\omega 2k) \\ 0 & 1 - \cos^2(\omega k) & \cos(\omega k) - \cos(\omega k)\cos(\omega 2k) \\ 0 & \cos(\omega k) - \cos(\omega 2k)\cos(\omega k) & sin^2(\omega 2k) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos(\omega k) & \cos(\omega 2k) \\ 0 & sin^2(\omega k) & 2sin^2(\omega k)\cos(\omega k) \\ 0 & 2sin^2(\omega k)\cos(\omega k) & sin^2(\omega 2k) \end{bmatrix} \rightarrow$$

$$s_3 - 2\cos(\omega 2k)\, s_2 \begin{bmatrix} 1 & \cos(\omega k) & \cos(\omega 2k) \\ 0 & sin^2(\omega k) & sin(\omega 2k)\, sin(\omega k) \\ 0 & 0 & sin^2(\omega 2k) - 4sin^2(\omega k)\cos^2(\omega k) \end{bmatrix}$$

$$sin^2(\omega k)x_2 + \sin(\omega 2k)\sin(\omega k)\, x_3 = 0$$
$$sin^2(\omega k)x_2 + 2sin^2(\omega k)\cos(\omega k)x_3 = 0$$
$$x_2 = -2\cos(\omega k)x_3$$

$$x_1 = -\cos(\omega k)(-\cos(\omega k)\, x_3) - \cos(\omega 2k)x_3$$
$$= (2\cos^2(\omega k) - \cos(\omega 2k))\, x_3$$
$$= x_3$$

Note that A ≥ 0 and hence a corresponds to the smallest eigenvalue of R.

$$\begin{bmatrix} r_0 & r_k & r_{2k} \\ r_k & r_0 & r_k \\ r_{2k} & r_k & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ -2c \\ 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 \\ -2c \\ 1 \end{bmatrix}; c = \cos(\omega k)$$

$$r_0 - 2c\, r_k + r_{2k} = \sigma^2$$
$$r_k - 2c\, r_0 + r_k = -2\, c\, \sigma^2$$
$$r_{2k} - 2\, c\ r_k + r_0 = \sigma^2$$

$$r_k - 2\, c(\sigma^2 + 2c\ r_k - r_{2k}) + r_k = -2\, c\, \sigma^2$$
$$2\, r_k - 4\, c^2\ r_k + 2\, c\, r_{2k} = 0$$
$$2\, r_k\, c^2 - r_{2k}\, c - r_k = 0$$

$$c = \frac{r_{2k} + \sqrt{r_{2k}^2 + 8r_k^2}}{4\, r_k}$$

$$c = \frac{r_{2k} + \sqrt{r_{2k}^2 + 8r_k^2}}{4\, r_k};$$

Since $r_0 - \sigma^2 = 2\, c\, r_k - r_{2k} > 0$

3. MATERIAL AND METHODS

3.1 VARIANCE ANALYSIS OF PHD

3.1.1 DATA MODEL AND PHD METHOD

We consider the following data model consisting of a single real tone in noise:

$$x(n) = \alpha \cos(\omega n + \phi) + e(n), \qquad n = 0,1, \dots, N-1 \qquad (3.1)$$

Where the amplitude α and frequency ω∈(0,π) are unknown constants while phase ϕ is a random variable uniformly distributed in the interval (-π,π].The noise $e(n)$ is assumed to be a zero-mean white Gaussian process with unknown variance σ^2 and independent of the phase ϕ. N denotes the number of available data samples.

The PHD frequency estimator is given by

$$\widehat{\omega} = arc\cos\left(\frac{r_2 + \sqrt{r_2^2 + 8r_1^2}}{4\, r_1}\right) \qquad (3.2)$$

Where r_m is the sample covariance that has the form

$$r_k = \frac{1}{N-k}\sum_{n-1}^{N-k} x(n)x(n+k), \qquad k = 1, 2.$$

Are the unbiased autocorrelation function estimates calculated from $x(n)$.

3.1.2 EXACT PHD VARIANCE DEVELOPMENT

Let $\lambda = \cos(\hat{\omega})$ and define a quadratic function

$$f(\rho) = 2r_1\rho^2 - r_2\rho - r_1 \tag{3.3}$$

For which λ is one of the roots. For sufficiently large N and/or SNR, which is defined as SNR$= \alpha^2/(2\sigma^2)$, λ will be closed to $\cos(\omega)$ and we can use the following formula to derive the variance of λ [3.4](also see [3.5]):

$$var(\lambda) \approx \left.\frac{E\{f^2(\rho)\}}{(E\{f'(\rho)\})^2}\right|_{\rho=\cos(\omega)} \tag{3.4}$$

The relationship between the variance of $\hat{\omega}$ and the variance of λ is given as

$$var(\hat{\omega}) \approx \frac{var(\lambda)}{sin^2(\omega)} \tag{3.5}$$

Here E denotes the expectation operator and $f'(\rho)$ is the derivative of the $f(\rho)$ with respect to ρ. The derivations of the expectations in (3.4) are given as follows. From (3.3) we have

$$E\{f^2(\rho)\}|_{\rho=\cos(\omega)}$$

$$= \cos^2(2\omega)E\{r_1^2\} - 2\cos(2\omega)\,E\{r_1r_2\} + cos^2(\omega)E\{r_2^2\} \tag{3.6}$$

And

$$E\{f'(\rho)\}|_{\rho=\cos(\omega)} = 4\cos(\omega)\,E\{r_1\} - E\{r_2\} \tag{3.7}$$

We see that the values of $E\{r_1\}, E\{r_2\}, E\{r_1^2\}, E\{r_1r_2\}$ and $E\{r_2^2\}$ are needed. These are derived as (For the derivations see the point 3.2.2 where the general case k-PHD is consider).

$$E\{r_1\} = \frac{\alpha^2 \cos(\omega)}{2} \tag{3.8}$$

$$E\{r_2\} = \frac{\alpha^2 \cos(2\omega)}{2} \tag{3.9}$$

$$E\{r_1^2\} = \frac{\sigma^4}{N-1} + \frac{\alpha^4 cos^2(\omega)}{4} + \frac{\alpha^4 sin^2((N-1)\omega)}{8(N-1)^2 sin^2(\omega)} + \frac{\alpha^2\sigma^2}{N-1} + \frac{\alpha^2\sigma^2(N-2)\cos(2\omega)}{(N-1)^2} \tag{3.10}$$

$$E\{r_1 r_2\} = \frac{\alpha^4\, cos(\omega)cos(2\omega)}{4} + \frac{\alpha^4 sin((N-1)\omega)sin((N-2)\omega)}{8(N-1)(N-2)sin^2(\omega)} + \frac{\alpha^2\sigma^2 cos(\omega)}{N-1} + \frac{\alpha^2\sigma^2(N-3)cos(3\omega)}{(N-1)(N-2)} \tag{3.11}$$

$$E\{r_2^2\} = \frac{\alpha^4}{N-2} + \frac{\alpha^4 cos^2(2\omega)}{4} + \frac{\alpha^4 sin^2((N-1)\omega)}{8(N-2)^2 sin^2(\omega)} + \frac{\alpha^2\sigma^2(N-4)\cos(4\omega)}{(N-2)^2} \tag{3.12}$$

Substituting$(3.8) - (3.12)$ into (3.6) and (3.7) using (3.4) and (3.5) , after simplifications, give

$$var(\widehat{\omega}) = \frac{A(N,\omega)SNR^{-2} + B(N,\omega)SNR^{-1} + C(N,\omega)}{D^2(\omega)sin^2(\omega)} \tag{3.13}$$

The variance is a function of ω, N and SNR. İt can be senn that A, B and C are of order N^{-1}, N^{-2}, N^{-2} , respectively, while D is not a function of N.

We have A, B, C and D are given as

$$A(N,\omega) = \frac{(2N-3) + (N-1)\cos(2\omega) + (N-2)cos(4\omega)}{8(N-1)(N-2)}$$

$$B(N,\omega) = \frac{4(N-1)(N-2) + (N-4)\cos(2\omega) - 4(N-1)\cos(4\omega) - N\cos(6\omega)}{8(N-1)^2(N-2)^2}$$

$$C(N,\omega) = \frac{\big((N-1)sin((N-1)\omega)\big) + sin\big((N-3)\omega\big) - (N-2)sin((N-1)\omega))^2}{32(N-1)^2(N-2)^2 sin^2(\omega)}$$

$$D(\omega) = \frac{2 + \cos(2\omega)}{2}$$

The expression in (3.13) gives the PHD frequency variance as a simple function of SNR. For low values of SNR the first term (3.13) becomes dominant and the variance is inversely proportional to the second power of the SNR. For high values of SNR the last term in the numerator of (3.13) becomes dominant and the variance becomes independent of the SNR. This unusual constant behavior of the PHD variance with respect to the SNR for high SNR regimes has also been observed for the constant phase case [3.14].

The variance expression in [3.14] derived for the constant phase case depends on the phase ϕ of the sinusoid in a very complicated way; see [3.4, equation(3.7)]. İn contrast, our variance expression (3.13) derived for the random phase case is independent of ϕ, as expected; by assuming a random phase we get rid of all the possible phase dependent terms.

For $N \geq 1$, the terms A, B and C can be approximated as

$$\hat{A}(N, \omega) = \frac{cos^2(\omega) + cos^2(2\omega)}{4N}$$

$$\hat{B}(N, \omega) = \frac{1}{2N^2}$$

$$\hat{C}(N, \omega) = \frac{cos^2(N\omega)}{8N^2}$$

Respectively, and consequently a simple approximate expression of $var(\hat{\omega})$ in this case is

$$var(\hat{\omega}) = \frac{\tilde{A}(N, \omega) SNR^{-2} + \tilde{B}(N, \omega) SNR^{-1} + \tilde{C}(N, \omega)}{D^2(\omega) k^2 sin^2(\omega)} \qquad (3.14)$$

3.2 VARIANCE ANALYSIS OF k-PHD

3.2.1 DATA MODEL AND k-PHD METHOD

We consider the following data model consisting of a single real tone in noise:

$$x(n) = \alpha \cos(\omega n + \phi) + e(n), \qquad n = 0,1, \dots, N-1 \tag{3.15}$$

Where the amplitude α and frequency $\omega \in (0, \pi)$ are unknown constants while phase ϕ is a random variable uniformly distributed in the interval $(-\pi, \pi]$. The noise $e(n)$ is assumed to be a zero-mean white Gaussian process with unknown variance σ^2 and independent of the phase ϕ. N denotes the number of available data samples.

The k-PHD method is a two-step frequency estimation algorithm. In the first step of the method, a rough estimate $\widehat{\omega}_1$ of the frequency ω is calculated using the PHD frequency estimator, probably with a small value of N. In the second step of the method, first the cosine of a k-th multiple, where k> 1 is an integer, of the frequency ω, $\cos(\omega k)$, is estimated from the estimator ρ_k given by

$$\rho_k = \frac{r_{2k} + \sqrt{r_{2k}^2 + 8r_k^2}}{4\, r_k} \tag{3.16}$$

where r_m is the sample covariance that has the form

$$r_m = \frac{1}{N-m} \sum_{n-1}^{N-m} x(n)x(n+m), \qquad m = k, 2k. \tag{3.17}$$

Then, the estimate $\widehat{\omega}_k$ of the frequency ω is calculated as

$$\widehat{\omega}_k = \frac{1}{k}\left((-1)^{i-1} arc\cos(\rho_k) + \left\lfloor \frac{i}{2} \right\rfloor 2\pi\right) \tag{3.18}$$

Where

$$i = 1 + \left\lfloor \frac{k\omega_1}{\pi} \right\rfloor \tag{3.19}$$

And $\lfloor \cdot \rfloor$ denotes the largest integer smaller than $(\cdot)$ (See De Sabata et. al., 2007). Note that for k=1 the k-PHD method is equivalent to the PHD method.

We remark here that the two-step procedure employed in the k-PHD method is a general frequency estimation process. In the first step of the process, a rough estimate of the frequency can be calculated from any low complexity method (such as from an estimator $\rho_1 = \cos(\widehat{\omega}_1)$ using $\widehat{\omega}_1 = arc\,(\rho_1)$; for a number of such estimators and a comparative study of their performances, see, e.g., Uz and Dilaveroğlu, 2019). In the second step of the process, any estimator which is designed for estimating an integer multiple of the frequency can be used (see, e.g., Toma et. al., 2007).

3.2.2 EXACT k-PHD VARIANCE DEVELOPMENT

Let prove that

$$\cos(kw) = \frac{r_{2k} + \sqrt{{r_{2k}}^2 + 8{r_k}^2}}{4r_k}$$

$R = \begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix}$ This matrix has two Symmetric eigenvectors and one Skew symmetric eigenvector

$R(0) = 1$

Similarly, the matrix $P = \begin{bmatrix} r_0 & r_k & r_{2k} \\ r_k & r_0 & r_k \\ r_{2k} & r_k & r_0 \end{bmatrix}$ has two Symmetric eigenvectors and one Skew symmetric eigenvector

$P(0) = 1$

The general equation obtain is: $(\lambda_i - 1)^2 - r_{2k}(\lambda_i - 1) - 2{r_k}^2 = 0$

And the solution will be:

$$\Rightarrow \lambda_i = 1 + \frac{r_{2k}}{2} \pm \sqrt{\frac{{r_{2k}}^2}{4} + 2{r_k}^2}$$

$$\Rightarrow \lambda_i = 1 + \frac{r_{2k}}{2} \pm \sqrt{\frac{{r_{2k}}^2 + 8{r_k}^2}{4}}$$

$$\Rightarrow \lambda_i - 1 = \frac{r_{2k}}{2} \pm \sqrt{\frac{{r_{2k}}^2 + 8{r_k}^2}{4}}$$

$$\Rightarrow \lambda_i - 1 = \frac{r_{2k}}{2} \pm \frac{1}{2}\sqrt{{r_{2k}}^2 + 8{r_k}^2}$$

$$\Rightarrow \lambda_i - 1 = \frac{r_{2k}}{2} \pm \frac{1}{2}\sqrt{{r_{2k}}^2 + 8{r_k}^2}$$

$$\Rightarrow \lambda_i - 1 = \frac{r_{2k} \pm \sqrt{{r_{2k}}^2 + 8{r_k}^2}}{2}$$

We have $\lambda_i - 1 = 2\cos(kw) \cdot r_k$

$$\Rightarrow 2\cos(kw) \cdot r_k = \frac{r_{2k} \pm \sqrt{{r_{2k}}^2 + 8{r_k}^2}}{2}$$

$$\Rightarrow \cos(kw) = \frac{r_{2k} \pm \sqrt{{r_{2k}}^2 + 8{r_k}^2}}{4\, r_k}$$

Define a second-order polynomial

$$\rho_k = 2r_k\rho^2 - r_{2k}\rho - r_k \tag{3.20}$$

For which ρ_k equals to one of the roots. For sufficiently large N and/or SNR , ρ_k will be close to $\cos(k\omega)$ and we can use the following formula to derive the variance of ρ_k**,** which is denoted by $var(\rho_k)$ (Chan and So, 2003) (also see So et. al., 2013):

$$var(\rho_k) \approx \frac{E\{f^2(\rho)\}}{(E\{f'(\rho)\})^2}\bigg|_{\rho=\cos(k\omega)} \tag{3.21}$$

In addition, the relationship between the variance of $\hat{\omega}_k$, which is denoted by $var(\hat{\omega}_k)$ and $var(\rho_k)$ is given as (Papoulis, 1991)

$$var(\hat{\omega}_k) \approx \frac{var(\rho_k)}{k^2 sin^2(k\omega)} \tag{3.22}$$

Here E denotes the expression operator and $f'(\rho)$ is the derivative of $f(\rho)$ with respect to ρ. The derivations of the expectations in (3.21) are given as follows. From (3.20) we have

$$E\{f^2(\rho)\}|_{\rho=\cos(k\omega)} = cos^2(2k\omega)E\{{r_k}^2\} - 2\cos(k\omega)\cos(2k\omega)\,E\{r_k r_{2k}\} + cos^2(k\omega)E\{{r_{2k}}^2\} \quad (3.23)$$

and

$$E\{f'(\rho)\}|_{\rho=\cos(k\omega)} = 4\cos(k\omega)\,E\{r_k\} - E\{r_{2k}\} \quad (3.24)$$

We see that the values of $E\{r_k\}$, $\{r_{2k}\}$, $E\{{r_k}^2\}$, $E\{r_k r_{2k}\}$, $E\{{r_{2k}}^2\}$ are required. These are derived as

$$* \; r_k = \frac{1}{N-K}\sum_{n=1}^{N-K} x(n)x(n+k)$$

$$* \; E[r_k] = \frac{1}{N-K}\sum_{n=1}^{N-K} E[x(n)x(n+k)]$$

$$x(n) = \alpha\cos(\omega n + \phi) + \omega(n) = s(n) + \omega(n)$$

$$E[x(n)x(n+k)] = E\{[s(n)+\omega(n)][s(n+k)+\omega(n+k)]\}$$
$$= s(n)s(n+k)$$
$$= \alpha^2\cos(\omega n+\phi)\cos(\omega(n+k)+\phi)$$
$$= \frac{\alpha^2}{2}[\cos(\omega k) + \cos(\omega(2n+k)+2\phi)]$$

$$\Rightarrow E[r_k] = \frac{1}{N-K}\sum_{n=1}^{N-K}\frac{\alpha^2}{2}[\cos(\omega k) + \cos(\omega(2n+k)+2\phi)]$$
$$= \frac{\alpha^2\cos(\omega k)}{2} + \frac{\alpha^2}{2(N-K)}\sum_{n=1}^{N-K}\cos(\omega(2n+k) + 2\phi)$$

$$* r_{2k} = \frac{1}{N-2K}\sum_{n=1}^{N-2K} x(n)x(n+2k)$$

$$* E[r_{2k}] = \frac{1}{N-2K}\sum_{n=1}^{N-2K} E[x(n)x(n+2k)]$$

$$* E[x(n)x(n+2k)] = E\{[s(n)+\omega(n)][s(n+2k)+\omega(n+2k)]\}$$

$$= s(n)s(n+k)$$

$$= \alpha^2 \cos(\omega n+\phi)\cos(\omega(n+2k)+\phi)$$

$$= \frac{\alpha^2}{2}[\cos(\omega 2k)+\cos(\omega(2n+2k)+2\phi)]$$

$$\Rightarrow E[r_{2k}] = \frac{\alpha^2\cos(\omega 2k)}{2}$$

$$+\frac{\alpha^2}{2(N-2K)}\sum_{n=1}^{N-2K}\cos(\omega(2n+2k)$$

$$+2\phi)$$

$$* {r_k}^2 = \frac{1}{(N-K)^2}\sum_{n=1}^{N-K}\sum_{m=1}^{N-K} x(n)x(n+k)x(m)x(m+k)$$

$$* E[x(n)x(n+k)x(m)x(m+k)]$$

$$= E\{[s(n)+w(n)][s(n+k)+\omega(n+k)][s(m)+\omega(m)][s(m+k)$$

$$+\omega(m+k)]\}$$

$$= E\{[s(n)s(n+k)+s(n)\omega(n+k)+s(n+k)\omega(n)+\omega(n)\omega(n$$

$$+k)][s(m)s(m+k)+s(m)\omega(m+k)+s(m+k)\omega(m)$$

$$+\omega(m)\omega(m+k)]\}$$

$$= s(n)s(n+k)s(m)s(m+k)+\sigma^2 s(n)s(m)\delta[n-m]$$

$$+\sigma^2 s(n)s(m+k)\delta[n+k-m]+\sigma^2 s(n+k)s(m)\delta[n-(k+m)]$$

$$+\sigma^2 s(n+k)s(m+k)\delta[n-m]+\sigma^4\,\delta[n-m]$$

$$* E[{r_k}^2] = \frac{1}{(N-K)^2}\sum_{n=1}^{N-K}\sum_{m=1}^{N-K} E[x(n)x(n+k)x(m)x(m+k)]$$

$$= \frac{1}{(N-K)^2}\sum_{n=1}^{N-K}\sum_{m=1}^{N-K} s(n)s(n+k)s(m)s(m+k)$$

$$+ \frac{\sigma^2}{(N-K)^2}\sum_{n=1}^{N-K}[s^2(n)+s^2(n+k)] + \frac{\sigma^2}{(N-K)^2}\sum_{n=1}^{N-2K} s(n)s(n+2k)$$

$$+ \frac{\sigma^2}{(N-K)^2}\sum_{m=1}^{N-2K} s(m)s(m+2k) + \frac{\sigma^4}{(N-K)^2}$$

$$\sum_{n=1}^{N-K}\sum_{m=1}^{N-K} s(n)s(n+k)s(m)s(m+k) = \left[\sum_{n=1}^{N-K} s(n)s(n+k)\right]^2$$

$$= \left\{\sum_{n=1}^{N-K}\frac{\alpha^2}{2}[\cos(\omega k)+\cos(\omega(2n+k)+2\phi)]\right\}^2$$

$$= \frac{\alpha^4}{4}[(N-K)\cos(\omega k)+g(N-K,K,\omega,\phi)]^2$$

$$\sum_{n=1}^{N-K}[s^2(n)+s^2(n+k)] = \sum_{n=1}^{N-K}\alpha^2 cos^2(\omega n+\phi)+\alpha^2 cos^2(\omega(n+k)+\phi)$$

$$= \alpha^2\sum_{n=1}^{N-K}\left\{\frac{1}{2}[1+\cos(\omega 2n+2\phi)]+\frac{1}{2}[1+\cos(\omega(2n+2k)+2\phi)]\right\}$$

$$\sum_{n=1}^{N-K}[s^2(n)+s^2(n+k)]$$

$$= \frac{\alpha^2}{2}[(N-K)+g(N-K,0,\omega,\phi)]+(N-K)+g(N-K,2k,\omega,\phi)$$

$$\sum_{n=1}^{N-2K} s(n)s(n+2k) = \sum_{n=1}^{N-2K}\alpha^2\cos(\omega n+\phi)\cos(\omega(n+2k)+\phi)$$

$$= \frac{\alpha^2}{2}\sum_{n=1}^{N-2K}[\cos(\omega 2k)+\cos(\omega(2n+2k)+2\phi)]$$

$$= \frac{\alpha^2}{2}[(N-2k)\cos(\omega 2k)+g(N-2k,k,\omega,\phi)]$$

$$\Rightarrow E[{r_k}^2]$$

$$= \frac{\sigma^4}{N-K} + \frac{\alpha^4[(N-k)\cos(\omega k) + g(N-k,k,\omega,\phi))]^2}{4(N-k)^2}$$

$$+ \frac{\alpha^2\sigma^2[(N-k) + g(N-k,0,\omega,\phi)]}{2(N-k)^2} + \frac{\alpha^2\sigma^2[(N-k) + g(N-k,2k,\omega,\phi)]}{2(N-k)^2}$$

$$+ \frac{\alpha^2\sigma^2[(N-2k)\cos(\omega 2k) + g(N-2k,2k,\omega,\phi)]}{(N-k)^2}$$

$$* \; r_k r_{2k} = \frac{1}{(N-k)(N-2k)} \sum_{n=1}^{N-K} \sum_{m=1}^{N-2K} x(n)x(n+k)x(m)x(m+2k)$$

$$* \; E[x(n)x(n+k)x(m)x(m+2k)]$$

$$= E\{[s(n) + \omega(n)][s(n+k) + \omega(n+k)][s(m) + \omega(m)][s(m+2k) + \omega(m+2k)]\}$$

$$= E\{[s(n)s(n+k) + s(n)\omega(n+k) + s(n+k)\omega(n) + \omega(n)\omega(n+k)][s(m)s(m+2k) + s(m)\omega(m+2k) + s(m+2k)\omega(m) + \omega(m)\omega(m+2k)]\}$$

$$= s(n)s(n+k)s(m)s(m+2k) + \sigma^2 s(n)s(m)\delta[n-(m+k)] + \sigma^2 s(n)s(m+2k)\delta[(n+k)-m] + \sigma^2 s(n+k)s(m)\delta[n-(m+2k)] + \sigma^2 s(n+k)s(m+2k)\delta[n-m]$$

$$* \; E[r_k r_{2k}] = \frac{1}{(N-k)(N-2k)} \sum_{n=1}^{N-K} \sum_{m=1}^{N-2K} s(n)s(n+k)s(m)s(m+2k)$$

$$+ \frac{\sigma^2}{(N-k)(N-2k)} \sum_{m=1}^{N-2K} s(m)s(m+k)$$

$$+ \frac{\sigma^2}{(N-k)(N-2k)} \sum_{n=1}^{N-3K} s(n)s(n+3k)$$

$$+ \frac{\sigma^2}{(N-k)(N-2k)} \sum_{m=1}^{N-3K} s(m)s(m+3k)$$

$$+ \frac{\sigma^2}{(N-k)(N-2k)} \sum_{n=1}^{N-2K} s(n+k)s(n+2k)$$

$$\sum_{n=1}^{N-K}\sum_{m=1}^{N-2K} s(n)s(n+k)s(m)s(m+2k) = \left[\sum_{n=1}^{N-K} s(n)s(n+k)\right]\left[\sum_{m=1}^{N-2K} s(m)s(m+2k)\right]$$

$$= \left\{\sum_{n=1}^{N-K}\frac{\alpha^2}{2}[\cos(\omega k) + \cos(\omega(2n+k)+2\emptyset)]\right\}\left\{\sum_{m=1}^{N-2K}\frac{\alpha^2}{2}[\cos(\omega 2k) + \cos(\omega(2n+2k)+2\emptyset)]\right\}s$$

$$= \frac{\alpha^4}{4}[(N-k)\cos(\omega k) + g(N-k,k,\omega,\phi)][(N-2k)\cos(\omega 2k) + g(N-2k,2k,\omega,\phi)]$$

$$\sum_{m=1}^{N-2K} s(m)s(m+k) = \sum_{m=1}^{N-2K}\frac{\alpha^2}{2}[\cos(\omega k) + \cos(\omega(2m+k)+2\phi)]$$

$$= \frac{\alpha^2}{2}[(N-2k)\cos(\omega k) + g(N-2k,k,\omega,\phi)]$$

$$\sum_{n=1}^{N-3K} s(n)s(n+3k) = \sum_{n=1}^{N-3K}\alpha^2\cos(\omega n+\phi)\cos(\omega(n+3k)+\phi)$$

$$= \sum_{n=1}^{N-3K}\frac{\alpha^2}{2}[\cos(\omega 3k) + \cos(\omega(2n+3k)+2\phi)]$$

$$= \frac{\alpha^2}{2}[(N-3k)\cos(\omega 3k) + g(N-3k,3k,\omega,\phi)]$$

$$\sum_{n=1}^{N-2K} s(n+k)s(n+2k) = \sum_{n=1}^{N-2K}\alpha^2\cos(\omega(n+k)+\phi)\cos(\omega(n+2k)+\phi)$$

$$= \sum_{n=1}^{N-2K}\frac{\alpha^2}{2}[\cos(\omega k) + \cos(\omega(2n+3k)+2\phi)]$$

$$= \frac{\alpha^2}{2}[(N-2k)\cos(\omega k) + g(N-2k,3k,\omega,\phi)]$$

$$\Rightarrow E[r_k r_{2k}]$$

$$= \frac{\alpha^4\,[(N-k)\ cos(\omega k) + g(N-k,k,\omega,\phi)][(N-2k)\ cos(\omega 2k) + g(N-2k,2k,\omega,\phi)]}{4(N-k)(N-2k)}$$

$$+ \frac{\alpha^2\sigma^2[(N-2k)\cos(\omega k) + g(N-2k,k,\omega,\phi)]}{2(N-k)(N-2k)}$$

$$+ \frac{\alpha^2\sigma^2[(N-3k)\cos(\omega 3k) + g(N-3k,3k,\omega,\phi)]}{(N-k)(N-2k)}$$

$$+ \frac{\alpha^2\sigma^2[(N-2k)\cos(\omega k) + g(N-2k,3k,\omega,\phi)]}{2(N-k)(N-2k)}$$

And

$$* {r_{2k}}^2 = \frac{1}{(N-2k)^2}\sum_{n=1}^{N-2K}\sum_{m=1}^{N-2K} x(n)x(n+2k)x(m)x(m+2k)$$

$$* E[x(n)x(n+2k)x(m)x(m+2k)]$$

$$= E\{[s(n)+\omega(n)][s(n+2k)+\omega(n+2k)][s(m)+\omega(m)][s(m+2k)+\omega(m+2k)]\}$$

$$= E\{[s(n)s(n+2k)+s(n)\omega(n+2k)+s(n+2k)\omega(n)+\omega(n)\omega(n+2k)][s(m)s(m+2k)+s(m)\omega(m+2k)+s(m+2k)\omega(m)+\omega(m)\omega(m+2k)]\}$$

$$= s(n)s(m+2k)s(m)s(m+2k) + \sigma^2 s(n)s(m)\delta[n-m] + \sigma^2 s(n)s(m+2k)\delta[(n+2k)-m] + \sigma^2 s(n+2k)s(m)\delta[n-(m+2k)] + \sigma^2 s(n+2k)s(m+2k)\delta[n-m] + \sigma^4\delta[n-m]$$

$$* E[{r_{2k}}^2] = \frac{1}{(N-2k)^2}\sum_{n=1}^{N-2K}\sum_{m=1}^{N-2K} s(n)s(n+2k)s(m)s(m+2k) + \frac{\sigma^2}{(N-2k)^2}\sum_{n=1}^{N-2K}[s^2(n)+s^2(n+2k)]$$

$$= \frac{\sigma^2}{(N-2k)^2}\sum_{n=1}^{N-4K} s(n)s(n+4k) + \frac{\sigma^2}{(N-2k)^2}\sum_{m=1}^{N-4K} s(m)s(m+4k) + \frac{\sigma^4}{(N-2k)}$$

$$\sum_{n=1}^{N-2K}\sum_{m=1}^{N-2K} s(n)s(n+2k)s(m)s(m+2k) = \left[\sum_{n=1}^{N-2K} s(n)s(n+2k)\right]^2$$

$$= \left\{\sum_{n=1}^{N-2K} \frac{\alpha^2}{2}[\cos(\omega 2k) + \cos(\omega(2n+2k)+2\phi)]\right\}^2$$

$$= \frac{\alpha^4}{4}[(N-2k)\cos(\omega 2k) + g(N-2k, 2k, \omega, \phi)]^2$$

$$\sum_{n=1}^{N-2K}[s^2(n) + s^2(n+2k)] = \sum_{n=1}^{N-2K}[\alpha^2 cos^2(\omega n+\phi) + \alpha^2 cos^2(\omega(n+2k)+\phi)]$$

$$= \sum_{n=1}^{N-2K} \alpha^2\left[\frac{1+cos(\omega 2n+2\phi)}{2} + \frac{1+\cos(\omega(2n+4k)+2\phi)}{2}\right]$$

$$= \frac{\alpha^2}{2}[(N-2k) + g(N-2k, 0, \omega, \phi) + (N-2k) + g(N$$
$$- 2k, 4k, \omega, \phi)]$$

$$\sum_{n=1}^{N-4K} s(n)s(n+4k) = \sum_{n=1}^{N-4K} \alpha^2\cos(\omega n+\phi)\cos(\omega(n+4k)+\phi)$$

$$= \frac{\alpha^2}{2}\sum_{n=1}^{N-4K} \cos(\omega 4k) + \cos(\omega(2n+4k)+2\phi)$$

$$= \frac{\alpha^2}{2}[(N-4k)\cos(\omega 4k) + g(N-4k, 4k, \omega, \phi)]$$

$$\Rightarrow E[{r_{2k}}^2] = \frac{\alpha^4[(N-2k)\cos(\omega 2k) + g(N-2k, 2k, \omega, \phi)]^2}{4(N-2k)^2}$$
$$+ \frac{\alpha^2\sigma^2[(N-2k) + g(N-2k, 0, \omega, \phi)]}{2(N-2k)^2}$$
$$+ \frac{\alpha^2\sigma^2[(N-2k) + g(N-2k, 4k, \omega, \phi)]}{2(N-2k)^2}$$
$$+ \frac{\alpha^2\sigma^2[(N-4k)\cos(\omega 4k) + g(N-4k, 4k, \omega, \phi)]}{(N-2k)^2}$$

$$* \; g(N, M, \omega, \phi) = \sum_{n=1}^{N} \cos(\omega(2n + M) + 2\phi)$$

$$\sum_{n=1}^{N} e^{j(\omega(2n+M)+2\phi)} = e^{j(\omega M+2\phi)} \sum_{n=1}^{N} e^{j\omega 2N}$$

$$= e^{j(\omega M+2\phi)} \sum_{n=0}^{N-1} e^{j\omega 2(n+1)} = e^{j(\omega(M+2)+2\phi)} \sum_{n=0}^{N-1} e^{j\omega 2n}$$

$$= e^{j(\omega(M+2)+2\emptyset)} \frac{1 - e^{j\omega 2N}}{1 - e^{j\omega 2}} = e^{j(\omega(M+2)+2\emptyset)} \frac{e^{j\omega N}}{e^{j\omega}} \frac{e^{-j\omega N} - e^{j\omega N}}{e^{-jw} - e^{j\omega}}$$

$$= \frac{sin(\omega N)}{sin(\omega)} e^{j(\omega(N+M+1)+2\emptyset)}$$

$$\sum_{n=1}^{N} cos(\omega(2n + M) + 2\phi) = \frac{sin(\omega N)\cos(\omega(N + M + 1) + 2\phi)}{sin(\omega)}$$

$$\Rightarrow E\{ r_k\} = \frac{\alpha^2 \cos(k\omega)}{2} \tag{3.25}$$

$$\Rightarrow E\{ r_{2k}\} = \frac{\alpha^2 \cos(2k\omega)}{2} \tag{3.26}$$

$$* \; Er_k{}^2 = \frac{\sigma^4}{N - k} + \frac{\alpha^4 cos^2(k\omega)}{4} + \frac{\alpha^4 E\, g^2(N - k, k, \omega, \emptyset)}{4(N - k)^2} + \frac{\alpha^2\sigma^2}{2(N - k)} + \frac{\alpha^2\sigma^2}{2(N - k)}$$

$$+ \frac{\alpha^2\sigma^2(N - 2k)\cos(2k\omega)}{(N - k)^2}$$

$$* \; E\, g^2(N - k, k, \omega, \phi) = E \frac{sin^2\big(\omega(N - k)\big)}{sin^2 w} \frac{1}{2}[1 + \cos(\omega(2N + 2) + 4\phi)]$$

$$= \frac{sin^2\big(\omega(N - k)\big)}{2\, sin^2\omega}$$

$$\Rightarrow E\{r_k{}^2\} = \frac{\sigma^4}{N - k} + \frac{\alpha^4 cos^2(k\omega)}{4} + \frac{\alpha^4 sin^2((N - k)\omega)}{8(N - k)^2 sin^2\omega} + \frac{\alpha^2\sigma^2}{N - k}$$

$$+ \frac{\alpha^2\sigma^2(N - 2k)\cos(2k\omega)}{(N - k)^2} \tag{3.27}$$

$$* \; Er_k r_{2k} = \frac{\alpha^4}{4}\cos(k\omega)\cos(2k\omega) + \frac{\alpha^4 E\, g(N - k, k, \omega, \phi) g(N - 2k, 2k, \omega, \phi)}{4(N - k)(N - 2k)}$$

$$+\frac{\alpha^2\sigma^2\cos(k\omega)}{2(N-k)}+\frac{\alpha^2\sigma^2(N-3k)\cos(3k\omega)}{(N-k)(N-2k)}+\frac{\alpha^2\sigma^2\cos(k\omega)}{2(N-k)}$$

$$* \; E\; g(N-k,k,\omega,\phi)g(N-2k,2k,\omega,\phi)$$

$$= E\frac{sin(\omega(N-k))cos(\omega(N+1)+2\phi)}{sin\,\omega}\frac{sin(\omega(N-2k))cos(\omega(N+1)+2\phi)}{sin\,\omega}$$

$$= E\frac{sin(\omega(N-k))sin(\omega(N-2k))}{sin^2\,\omega}\,cos^2(\omega(N+1)+2\phi)$$

$$= E\frac{sin(\omega(N-k))sin(\omega(N-2k))}{sin^2\,\omega}\,\frac{1}{2}[1+\cos(\omega(2N+2)+4\phi)]$$

$$=\frac{sin(\omega(N-k))sin(\omega(N-2k))}{2\,sin^2\,\omega}$$

$$\Rightarrow Er_kr_{2k}=\frac{\alpha^4\cos(k\omega)\cos(2k\omega)}{4}+\frac{\alpha^4 sin(\omega(N-k))sin(\omega(N-2k))}{8(N-k)(N-2k)\,sin^2\,\omega}$$

$$+\frac{\alpha^2\sigma^2\cos(k\omega)}{N-k}$$

$$+\frac{\alpha^2\sigma^2(N-3k)\cos(3k\omega)}{(N-k)(N-2k)} \qquad (3.28)$$

$$* \; {Er_{2k}}^2=\frac{\sigma^4}{N-2k}+\frac{\alpha^4cos^2(2k\omega)}{4}+\frac{\alpha^4E\;g^2(N-2k,2k,\omega,\phi)}{4(N-2k)^2}$$

$$+\frac{\alpha^2\sigma^2}{2(N-2k)}+\frac{\alpha^2\sigma^2}{2(N-2k)}+\frac{\alpha^2\sigma^2(N-4k)\cos(4k\omega)}{(N-2k)^2}$$

$$* \; E\;g^2(N-2k,2k,\omega,\phi)=E\frac{sin^2\big(\omega(N-2k)\big)}{sin^2\omega}cos^2(\omega(N+1)+2\phi)$$

$$=E\frac{sin^2\big(\omega(N-2k)\big)}{sin^2\omega}\frac{1}{2}[1+\cos(\omega(2N+2)+4\phi)]$$

$$=\frac{sin^2\big(\omega(N-2k)\big)}{2\,sin^2\omega}$$

$$\Rightarrow \; {Er_{2k}}^2=\frac{\sigma^4}{N-2k}+\frac{\alpha^4cos^2(2k\omega)}{4}+\frac{\alpha^4sin^2(\omega(N-2k))}{8(N-2k)^2\,sin^2\,\omega}\;+\frac{\alpha^2\sigma^2}{(N-2k)}$$

$$+\frac{\alpha^2\sigma^2(N-4k)\cos(4k\omega)}{(N-2k)^2} \qquad (3.29)$$

$$\text{let } \lambda=\cos(k\omega)=\frac{r_{2k+\sqrt{{r_{2k}}^2+8\,{r_k}^2}}}{4\,r_k}$$

and

$$\tilde{\lambda} = \cos(k\omega) = \frac{\hat{r}_{2k} + \sqrt{\hat{r}_{2k}^2 + 8\hat{r}_k^2}}{4\hat{r}_k}$$

$\tilde{\lambda}$ is one of the roots of $f(\tilde{\lambda}) = 2\hat{r}_k\tilde{\lambda}^2 - \hat{r}_{2k}\tilde{\lambda} - \hat{r}_k$

Assuming $\hat{\lambda}$ is sufficiently near λ

$0 = f(\tilde{\lambda}) \approx f(\lambda) + (\tilde{\lambda} - \lambda)f'(\lambda)$

Assuming $f'(\lambda)$ is sufficiently smooth around $\tilde{\lambda} = \lambda$

$f'(\lambda) \approx E[f'(\lambda)]$

We have $\tilde{\lambda} - \lambda \approx -\frac{f(\lambda)}{E[f'(\lambda)]}$

$\text{bias}(\tilde{\lambda}) = E[\tilde{\lambda}] - \lambda \approx -\frac{E[f(\lambda)]}{E[f'(\lambda)]}$

$E[f(\lambda)] = 2\lambda^2 E[\hat{r}_k] - E[\hat{r}_{2k}] - E[\hat{r}_k]$

$= (2\lambda^2 - 1)E[\hat{r}_k] - \lambda\, E[\hat{r}_{2k}]$

$= [2cos^2(k\omega) - 1]\frac{\alpha^2\cos(k\omega)}{2} - \cos(kw)\frac{\alpha^2\cos(2k\omega)}{2} = 0$

$\text{bias}(\tilde{\lambda}) \approx 0$

$f'(\tilde{\lambda}) = 4\hat{r}_k\tilde{\lambda} - \hat{r}_{2k}$

$E[f'(\lambda)] = 4\lambda E[\hat{r}_k] - E[\hat{r}_{2k}]$

$= 4\cos(k\omega)\frac{\alpha^2\cos(k\omega)}{2} - \frac{\alpha^2\cos(2k\omega)}{2}$

$= \frac{\alpha^2}{2}[4\,cos^2(k\omega) - \cos(2k\omega)]$

$= \frac{\alpha^2[2 + \cos(2k\omega)]}{2}$

$MSE(\tilde{\lambda}) = E[(\tilde{\lambda} - \lambda)^2] \approx \frac{E[f^2(\lambda)]}{(E[f'(\lambda)])^2}$

$f^2(\tilde{\lambda}) = [(2\tilde{\lambda}^2 - 1)\hat{r}_k - \tilde{\lambda}\hat{r}_{2k}]^2 = (2\tilde{\lambda}^2 - 1)^2\ \hat{r}_k{}^2 - 2\tilde{\lambda}(2\tilde{\lambda}^2 - 1)\hat{r}_k\hat{r}_{2k} + \tilde{\lambda}^2\hat{r}_{2k}{}^2$

$E[f^2(\lambda)] = cos^2(2k\omega)E[\hat{r}_k{}^2] - 2\cos(k\omega)\cos(2k\omega)E[\hat{r}_k\hat{r}_{2k}] + cos^2(k\omega)E[\hat{r}_{2k}{}^2]$

$$= \frac{\sigma^4}{N-K}cos^2(2k\omega) + \frac{\sigma^4}{N-2K}cos^2(k\omega) + \frac{\alpha^4\, cos^2(k\omega)}{4}cos^2(2k\omega)$$
$$+ \frac{\alpha^4 sin^2[(N-k)\omega]}{8(N-k)^2\, sin^2\omega}cos^2(2k\omega)$$
$$+ \frac{\alpha^4 \cos(k\omega)\cos(2k\omega)}{4}[-2\cos(k\omega)\cos(2k\omega)]$$
$$+ \frac{\alpha^4\, sin[(N-k)\omega]sin[(N-2k)\omega]}{8(N-k)(N-2k)sin^2\omega}[-2\cos(k\omega)\cos(2k\omega)]$$
$$+ \frac{\alpha^4\, cos^2(2k\omega)}{4}cos^2(k\omega) + \frac{\alpha^4 sin^2[(N-2k)\omega]}{8(N-2k)^2\, sin^2\omega}cos^2(2k\omega)$$
$$+ \frac{\alpha^2\sigma^2}{N-k}cos^2(2k\omega) + \frac{\alpha^2\sigma^2(N-2k)\cos(2k\omega)}{(N-k)^2}cos^2(2k\omega)$$
$$+ \frac{\alpha^2\sigma^2\, cos(k\omega)}{N-k}[-2cos(k\omega)cos(2k\omega)]$$
$$+ \frac{\alpha^2\sigma^2\, (N-3k)cos(3k\omega)}{(N-k)(N-2K)}[-2cos(k\omega)cos(2k\omega)]$$
$$+ \frac{\alpha^2\sigma^2}{N-2k}cos^2(k\omega) + \frac{\alpha^2\sigma^2(N-4k)\cos(4k\omega)}{(N-2k)^2}cos^2(k\omega)$$

$$\sigma^4: \frac{cos^2(2k\omega)}{N-k} + \frac{cos^2(k\omega)}{N-2k}$$
$$= \frac{(N-2K)cos^2(2k\omega) + (N-k)cos^2(k\omega)}{(N-k)(N-2k)}$$
$$= \frac{N[cos^2(2k\omega) + cos^2(k\omega)] - k[2cos^2(2k\omega) + cos^2(k\omega)]}{(N-k)(N-2k)}$$
$$= \frac{N\frac{1}{2}[2 + \cos(2k\omega) + \cos(4k\omega)] - k\frac{1}{2}[3 + \cos(2k\omega) + 2\cos(4k\omega)]}{(N-k)(N-2k)}$$
$$= \frac{(2N-3k) + (N-k)\cos(2k\omega) + (N-2k)\cos(4k\omega)}{2(N-k)(N-2k)}$$

$$\frac{\alpha^4}{4}: cos^2(k\omega)cos^2(2k\omega) - 2cos^2(k\omega)cos^2(2k\omega) + cos^2(k\omega)cos^2(2k\omega) = 0$$

$$\frac{\alpha^4}{8sin^2(w)}:\frac{sin^2[(N-k)\omega]cos^2(2k\omega)}{(N-k)^2}$$
$$-\frac{2sin[(N-k)\omega]sin[(N-2k)\omega]\cos(k\omega)\cos(2k\omega)}{(N-k)(N-2k)}$$
$$+\frac{sin^2[(N-2k)\omega]cos^2(k\omega)}{(N-2k)^2}$$
$$=(\frac{sin[(N-k)\omega]\cos(2k\omega)}{N-k}-\frac{sin[(N-2k)\omega]\cos(k\omega)}{N-2k})^2$$

$$=\frac{\{-(N-k)sin[(N-k)\omega]-k\,sin[(N-3k)\omega]+(N-2k)sin[(N+k)\omega]\}}{4(N-k)^2(N-2k)^2}$$

$$\alpha^2\sigma^2:\frac{cos^2(2k\omega)}{N-k}+\frac{(N-2k)cos^3(2k\omega)}{(N-k)^2}-\frac{2cos^2(k\omega)\cos(2k\omega)}{N-k}$$
$$-\frac{2(N-3k)\cos(k\omega)\cos(2k\omega)\cos(3k\omega)}{(N-k)(N-2k)}+\frac{cos^2(k\omega)}{N-2k}$$
$$+\frac{(N-4k)\cos(4k\omega)cos^2(k\omega)}{(N-2k)^2}$$
$$=\frac{[4k(N-k)(N-2k)+k^2(N-4k)\cos(2k\omega)-4k^2(N-k)\cos(4k\omega)-k^2Ncos(6k\omega)]}{4(N-k)^2(N-2k)^2}$$

Substituting$(3.25)-(3.29)$ into (3.23) and (3.24) using (3.21) and (3.22) , after simplifications, give

$$var(\widehat{\omega}_k)=\frac{A(N,k,\omega)SNR^{-2}+B(N,k,\omega)SNR^{-1}+C(N,k,\omega)}{D^2(k,\omega)k^2sin^2(k\omega)} \qquad (3.30)$$

$$E[f^2(\lambda)]=A'(N,k,\omega)\sigma^4+B'(N,k,\omega)\alpha^2\sigma^2+C(N,k,w)\alpha^4$$

Where

$$A'(N,k,\omega)=\frac{(2N-3k)+(N-k)\cos(2k\omega)+(N-2k)\cos(4k\omega)}{2(N-k)(N-2k)}$$

$$B'(N,k,\omega)$$
$$=\frac{[4k(N-k)(N-2k)+k^2(N-4k)\cos(2k\omega)-4k^2(N-k)\cos(4k\omega)-k^2Ncos(6k\omega)]}{4(N-k)^2(N-2k)^2}$$

$$A(N,k,\omega)=\frac{(2N-3k)+(N-k)\cos(2k\omega)+(N-2k)\cos(4k\omega)}{8(N-k)(N-2k)} \qquad (3.31)$$

$$B(N,k,\omega)=\left[\frac{4k(N-k)(N-2k)+k^2(N-4k)\cos(2k\omega)}{8(N-k)^2(N-2k)^2}\right.$$
$$\left.-\frac{4k^2(N-k)\cos(4k\omega)+k^2N\cos(6k\omega)}{8(N-k)^2(N-2k)^2}\right] \quad (3.32)$$

$$C(N,k,\omega)$$
$$=\frac{\{(N-k)sin[(N-k)\omega]+k\,sin[(N-3k)\omega]-(N-2k)sin[(N+k)\omega]\}^2}{32(N-k)^2(N-2k)^2sin^2(\omega)} \quad (3.33)$$

$$D(k,\omega)=\frac{2+\cos(2k\omega)}{2} \quad (3.34)$$

$$(E[f'(\lambda)])^2=\alpha^4D^2(k,\omega)$$
$$var(\tilde{\lambda})\approx\frac{A'\sigma^4+B'\alpha^2\sigma^2+C\,\alpha^4}{\alpha^4\,D^2}$$
$$=\frac{(A'/4)}{D^2\eta^2}+\frac{(B'/2)}{D^2\eta^2}+\frac{C}{D^2}$$
$$=\frac{A(N,k,\omega)}{D^2(k,\omega)\eta^2}+\frac{B(N,k,\omega)}{D^2(k,\omega)\eta^2}+\frac{C(N,k,\omega)}{D^2(k,\omega)}$$
$$var(\hat{\omega})\approx\frac{var(\tilde{\lambda})}{k^2sin^2(k\omega)}$$

The variance is a function of w , N , k and SNR , which is defined as $SNR=\alpha^2/(2\sigma^2)$. İt can be seen that A , B and C are of order N^{-1}, N^{-2}, N^{-2}, respectively, while D is not a function of N.

The expression in (3.30) gives the variance of the k-PHD frequency estimator $\widehat{w_k}$ as a simple function of the SNR. For low $SNRs$, the term of (3.30) becomes dominant and $var(\widehat{w_k})$ is proportional to (N^{-1},SNR^{-2}); for high $SNRs$, the last term of (3.30) becomes dominant and the variance is proportional to N^{-2} and, interestingly, independent of SNR. This unusual constant behavior of the variance with respect to SNRat high $SNRs$ has also been observed for the constant phase case (De Sabata ET. Al., 2007).

The variance expression in De Sabata et.al.,(2007) derived for the constant phase case depends on the phase ϕ of the sinusoid in a very complicated way; see De Sabata et.al.(2007), equation (3.24). İn constant, the variance expression (3.30) derived for the random phase case is independent of, as expected; by assuming a random phase we got rid of all the possible phase dependent terms.

For $N \gg 1$, the terms A , B and C can be approximated as $N \to \infty$

$$\tilde{A}(N,k,\omega) = \frac{(2N-3k)+(N-k)\cos(2k\omega)+(N-2k)\cos(4k\omega)}{8(N-k)(N-2k)}$$

$$\approx \frac{2+\cos(2k\omega)+\cos(4k\omega)}{8N}$$

$$\tilde{A}(N,k,\omega) \approx \frac{cos^2(k\omega)+cos^2(2k\omega)}{4N} \qquad (3.35)$$

$$\tilde{B}(N,k,\omega)$$

$$= \frac{[4k(N-k)(N-2k)+k^2(N-4k)\cos(2k\omega)-4k^2(N-k)\cos(4k\omega)-k^2N\cos(6k\omega)]}{8(N-k)^2(N-2k)^2}$$

$$\tilde{B}(N,k,\omega) \approx \frac{k}{2N^2} \qquad (3.36)$$

$$\sqrt{2}\sqrt{C} = \frac{(N-k)\,sin[(N-k)\omega]+k\,sin[(N-3k)\omega]-(N-2k)sin[(N+k)\omega]}{(N-k)(N-2k)\sin(\omega)}$$

$$= \frac{N-k}{N-2k}\frac{sin[(N-k)\omega]}{(N-k)\sin(\omega)} + \frac{k(N-3k)}{(N-k)(N-2k)}\frac{sin[(N-3k)\omega]}{(N-3k)\sin(\omega)}$$

$$-\frac{N+k}{N-k}\frac{sin[(N-k)\omega]}{(N-k)\sin(\omega)}\quad\frac{sin[(N-k)\omega]}{(N-k)\sin(\omega)}$$

$$-\frac{sin[(N+k)\omega]}{(N+k)\sin(\omega)}$$

$$= \frac{(N+k)sin[(N-k)\omega]-(N-k)sin[(N+k)\omega]}{(N-k)(N+k)\sin(\omega)}$$

$$= \frac{N(sin[(N-k)\omega]-sin[(N+k)\omega])+k(sin[(N-k)\omega]+sin[(N+k)\omega])}{(N-k)(N+k)\sin(\omega)}$$

$$= \frac{-2Ncos(N\omega)\sin(k\omega) + 2k\sin(N\omega)\cos(k\omega)}{(N-k)(N+k)\sin(\omega)} \qquad \approx \frac{-2\cos(N\omega)\sin(k\omega)}{N\sin(\omega)}$$

$$\tilde{C}(N,k,\omega) \approx \frac{cos^2(N\omega)sin^2(k\omega)}{8N^2sin^2(\omega)} \tag{3.37}$$

And consequently a simple approximate expression of $var(\widehat{\omega}_k)$ in this case is

$$var(\widehat{\omega}_k) = \frac{\tilde{A}(N,k,\omega)SNR^{-2} + \tilde{B}(N,k,\omega)SNR^{-1} + \tilde{C}(N,k,\omega)}{D^2(k,\omega)k^2sin^2(k\omega)} \tag{3.38}$$

Note that the variance in (3.30) and (3.38) is unbounded at frequencies

$$\omega = \frac{i\,\pi}{k}\,, \quad i = 0,1,\dots,k. \tag{3.39}$$

In practice, a value of k can be chosen so that the first estimate $\widehat{\omega}_1$ is not close to a value given by (3.39).

4. RESULTS AND DISCUSSION

4.1 NUMERICAL EXAMPLES FOR PHD

Computer simulations had been performed in order to validate our theoretical results. The tone amplitude was assigned to 2 while different SNRs were obtained by properly scaling the noise samples. All simulation results were obtained by averaging 1000 independent runs.

Figure 4.1-4.3 show the variances of frequency of the PHD estimator for k=2 and 3 for $\omega \in [0.025\pi, 0.975\pi]$ at SNR=20 dB and for N=8, 32 and 128, respectively. The variance expressions of (3.13) and (3.14) were also included. We observe that the measured variances conformed the exact expression (3.13) even for small value of $N = 8$ provided that the frequency ω is not very near to 0 or π. Also, the simple approximate expression (3.14) , whose derivation assumed $N \geq 1$, was in a good agreement with the measured ones even for moderate values of N such as N =32.

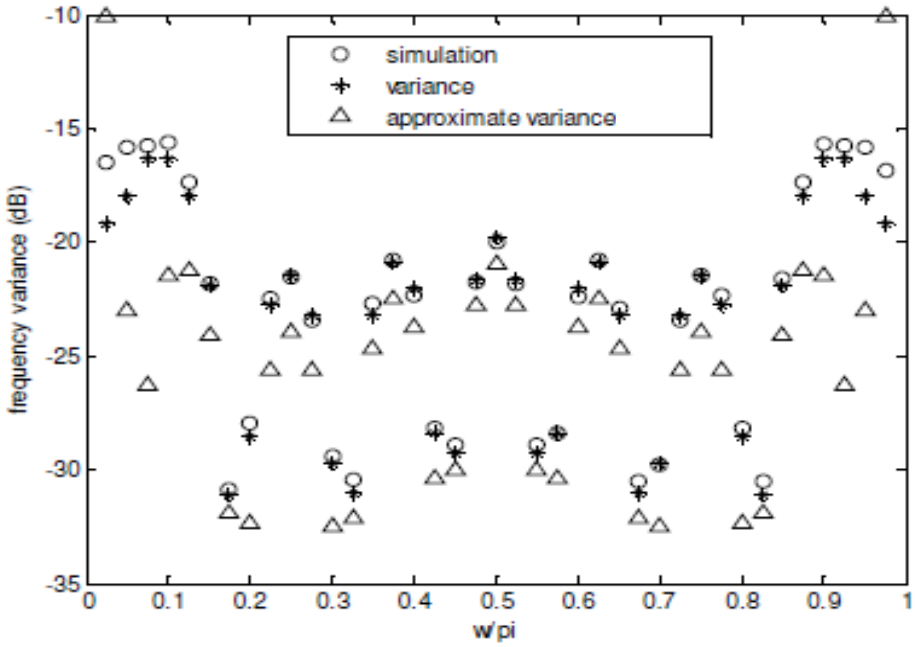


Figure 4.1. Variances of Frequency versus ω at SNR= 20 and $N = 8$.

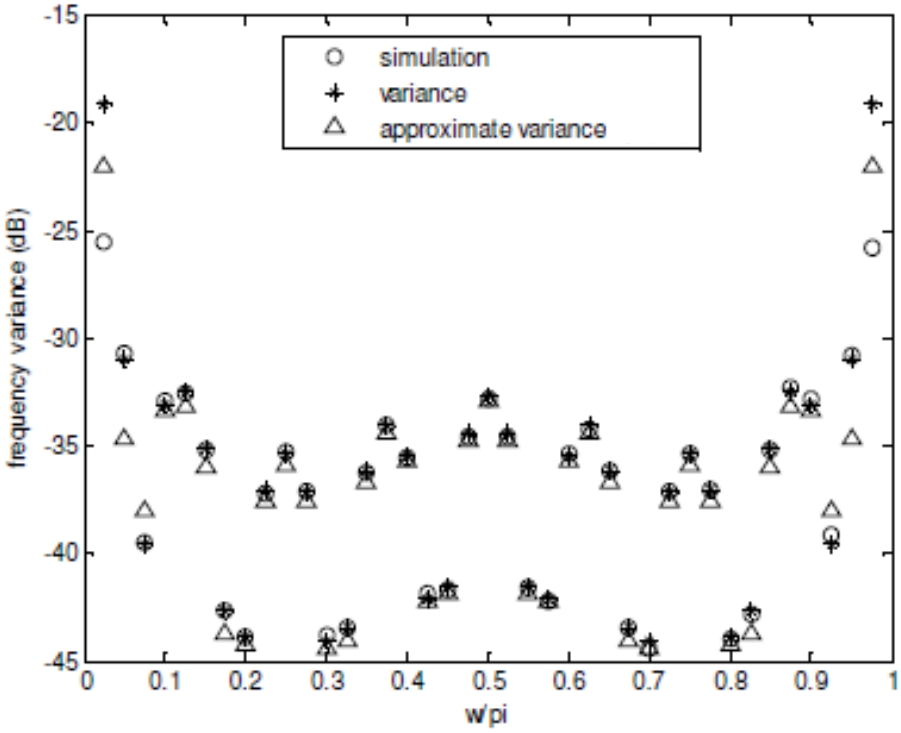


Figure 4.2. Variances of Frequency versus ω at SNR$= 20$ and $N = 32$.

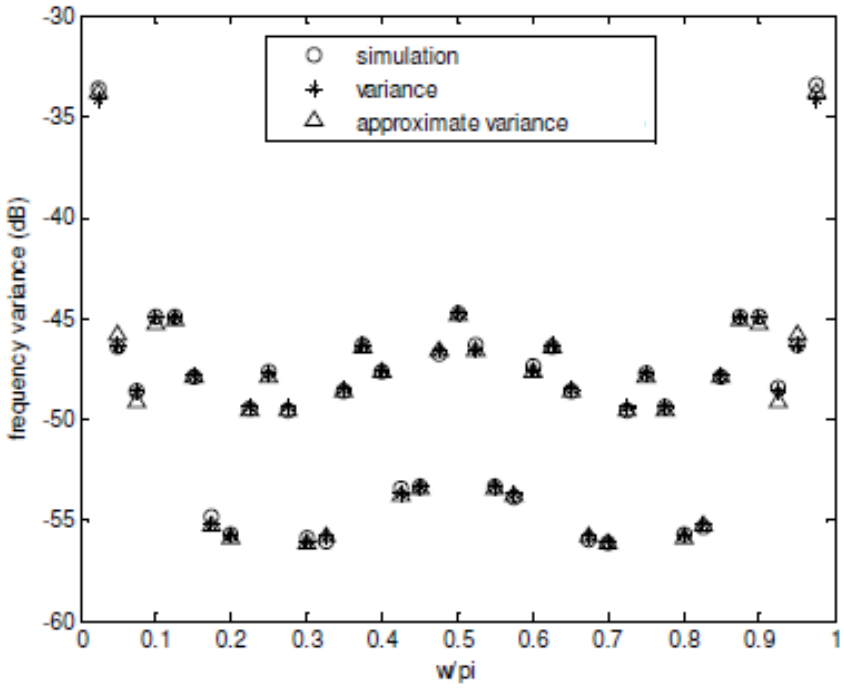


Figure 4.3. Variances of Frequency versus ω at SNR$= 20$ and $N = 128$.

Figure 4.4-4.6 show the Variances of Frequency versus SNR at $\omega = 3\pi/8$ and for $N = 8, 32$ and 128, respectively.

İt can be oserved that (1.31) agreed well with the simulation results provided that $(N \cdot \text{SNR})$ is not too small while (1.32) was a good approximation for $N \geq 32$.

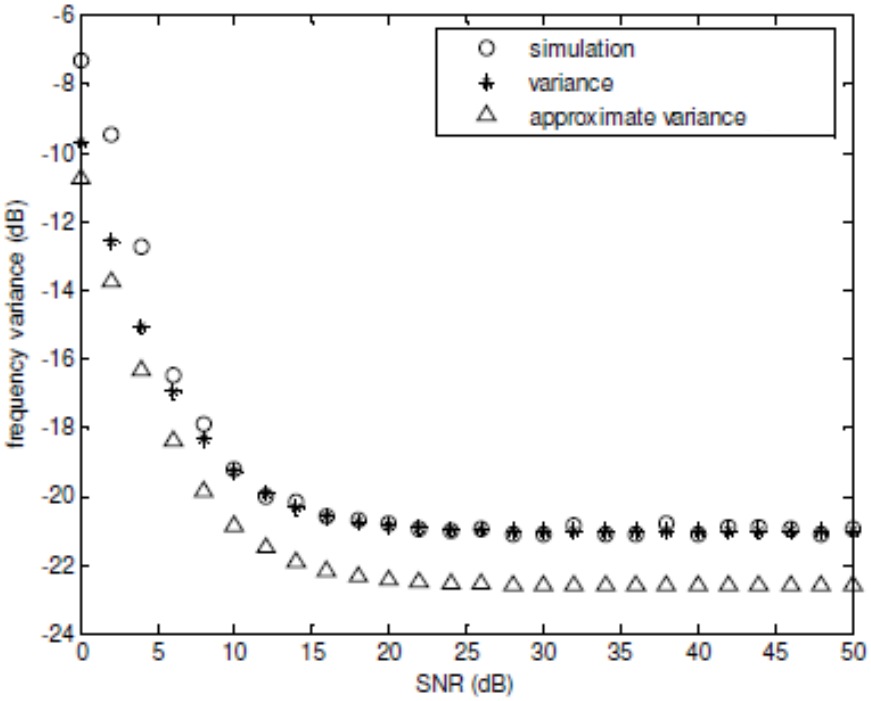


Figure 4.4. Variances of Frequency versus SNR at $\omega = 3\pi/8$ and $N = 8$.

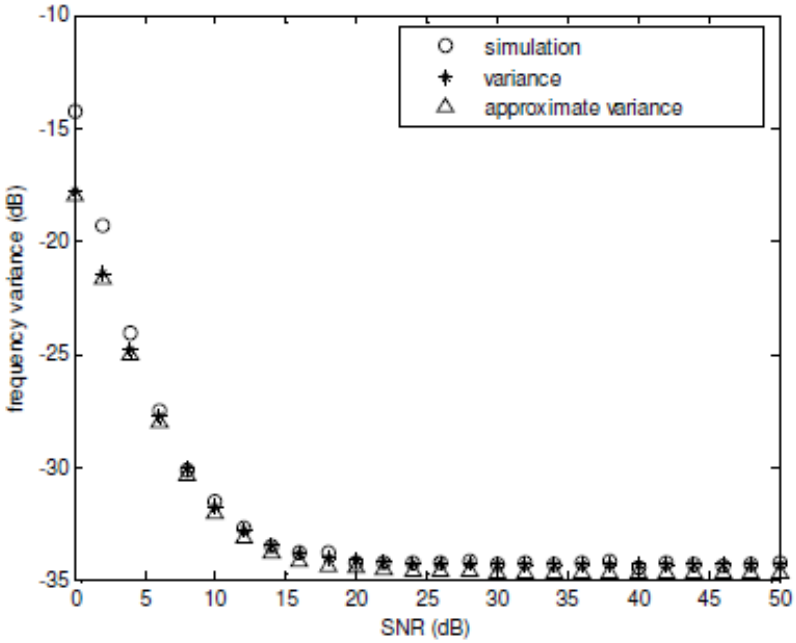


Figure 4.5. Variances of Frequency versus SNR at $\omega = 3\pi/8$ and $N = 32$.

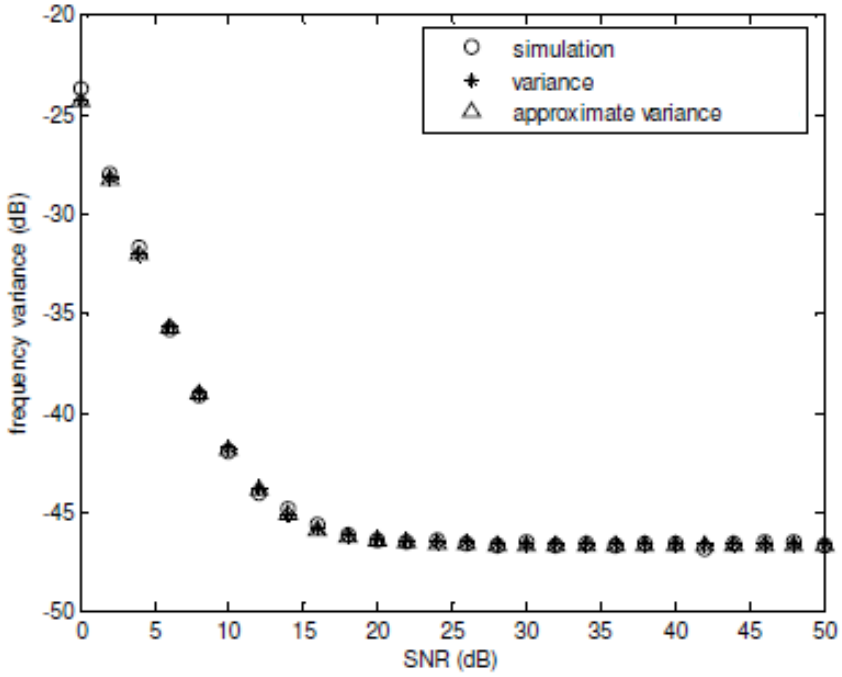


Figure 4.6. Variances of Frequency versus SNR at $\omega = \frac{3\pi}{8}$ and $N = 128$.

4.2 NUMERICAL EXAMPLES FOR k –PHD

Computer simulations had been performed in order to validate our theoretical results. The tone amplitude was assigned to 2 while different SNRs were obtained by properly scaling the noise samples. All simulation results were obtained by averaging 1000 independent runs.

Figure 4.7. show the Variances of Frequency of the PHD estimator and k-PHD estimators for k=2 and 3 for $\omega \in [0.025\pi, 0.975\pi]$ at SNR=20 dB and for N=32. It is seen that at every frequency the variance of the k-PHD for k=2 and/or for k=3 was smaller than that of the PHD for (k=1).

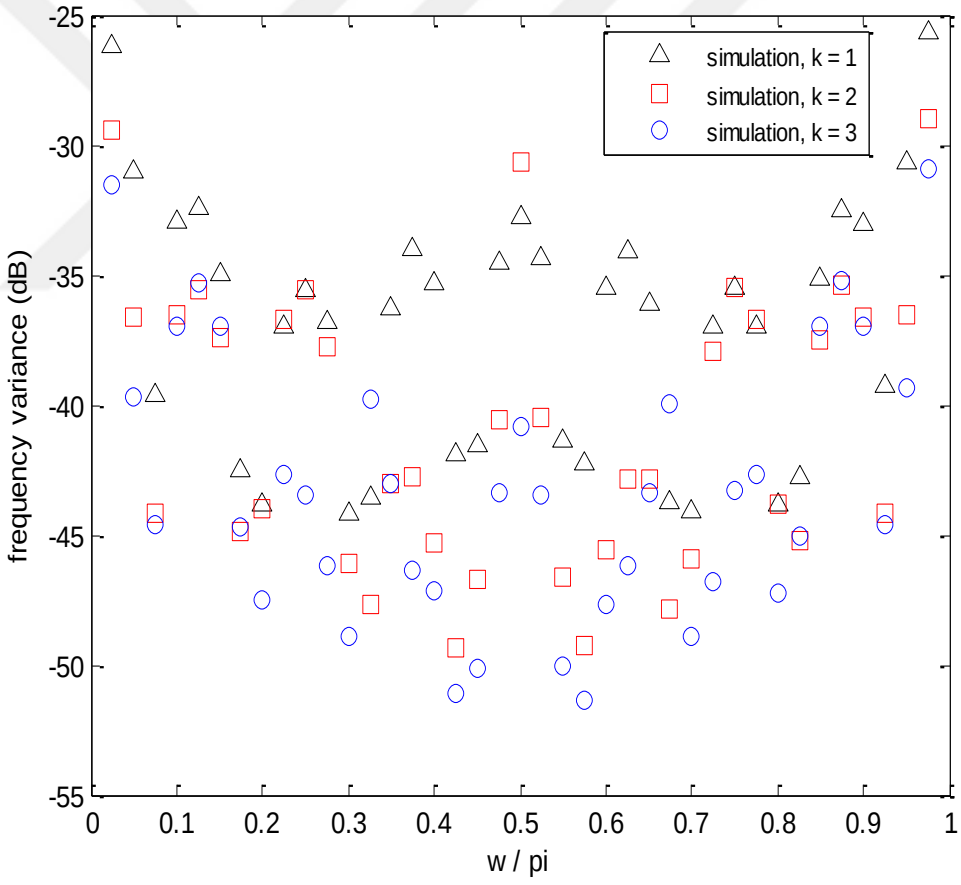


Figure 4.7. Variances of Frequency versus ω at SNR $= 20$ dB and N $= 32$.

Figure 4.8. Shows the Variances of Frequency of the PHD and the k-PHD for $k = 3$ versus w at $SNR = 20$ dB for $N = 128$. The variance expressions of (3.30) and (3.38) were also included. We observe that the measured variances conformed to the exact expression (3.30) provided that the frequency w is not very near to 0 or π. Also, the simple approximate expression (3.38) was in good agreement with the measured ones. The decrease in variances for the k-PHD as compared to the PHD is apparent from the Figure for the whole range of the frequency except at w=0.325π and w=0.675π as predicted by (3.39).

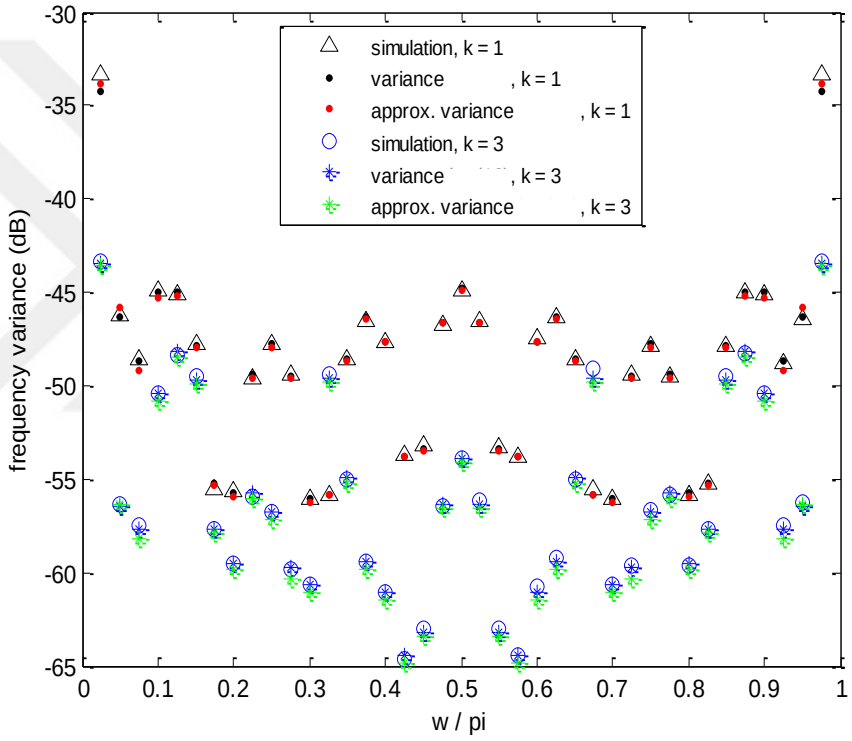


Figure 4.8. Variances of Frequency versus ω at SNR = 20 dB and N = 128

Figure 4.9 shows variances of frequency of the PHD method and the k-PHD method for k=4 versus SNRat w=3π/8 and for N=128. It can be observed that (3.30) agreed well with the simulation results while (3.38) was a good approximation provided that SNR is not too small.

The variances of the k-PHD were smaller than those of the PHD for the whole range of SNRand the reduction was more than 9 dB for SNR≥20 dB.

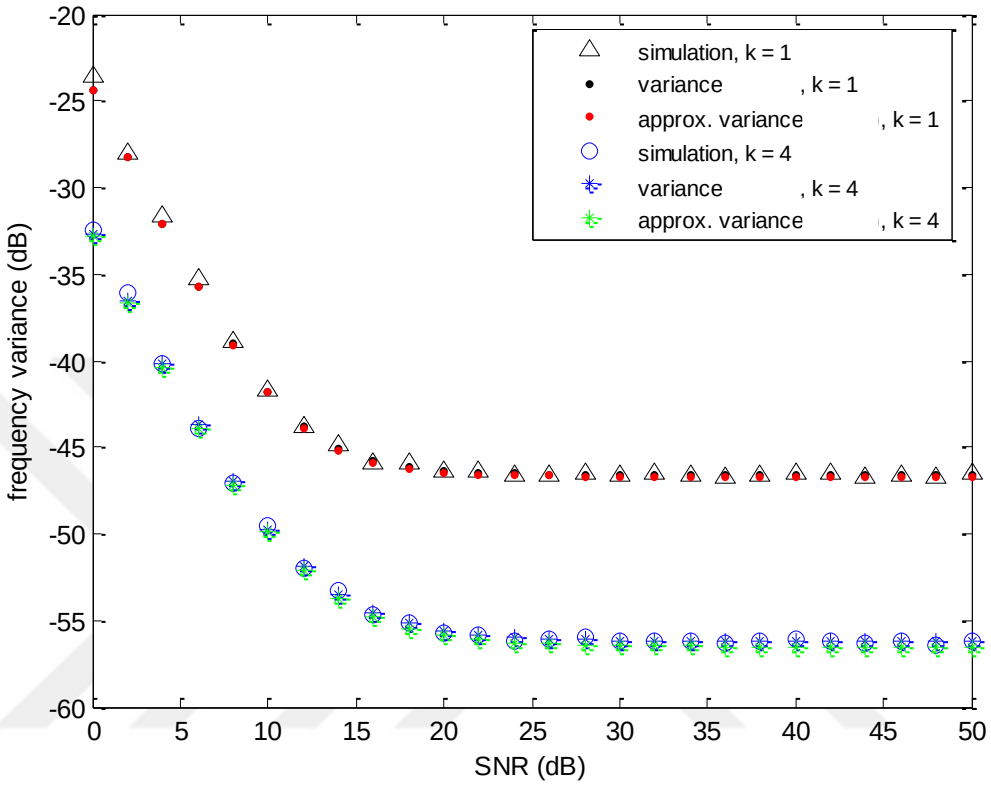


Figure 4.9. Variances of Frequency versus SNR at $\omega = \frac{3\pi}{8}$ and N=128.

Table 1 shows frequency standard deviations of the PHD and the k-PHD for $k = 3$ for different values of the data length N at $SNR = 20$ dB and $w = 3\pi/8$. It can be seen that (3.30) and (3.38) predicted the measured variances very well for all values of N and $N \geq 64$, respectively.

Table 4.1. Frequency standard deviations versus N at SNR = 20 dB and w = 3π/8.

	Measured standard deviation	**Standard deviation by (3.30)**	**Approximate standard deviation by (3.38)**
$N = 16, k = 1$	$4.1668\text{x}10^{-2}$	$4.1129\text{x}10^{-2}$	$3.7769\text{x}10^{-2}$
$k = 3$	$1.3547\text{x}10^{-2}$	$1.3158\text{x}10^{-2}$	$7.7352\text{x}10^{-3}$
$N = 32, k = 1$	$1.9294\text{x}10^{-2}$	$1.9703\text{x}10^{-2}$	$1.8903\text{x}10^{-2}$
$k = 3$	$4.6939\text{x}10^{-3}$	$4.9182\text{x}10^{-3}$	$3.8958\text{x}10^{-3}$
$N = 64, k = 1$	$9.8179\text{x}10^{-3}$	$9.6655\text{x}10^{-3}$	$9.4703\text{x}10^{-3}$
$k = 3$	$2.2516\text{x}10^{-3}$	$2.2024\text{x}10^{-3}$	$1.9758\text{x}10^{-3}$
$N = 128, k = 1$	$4.8010\text{x}10^{-3}$	$4.8020\text{x}10^{-3}$	$4.7538\text{x}10^{-3}$
$k = 3$	$1.0805\text{x}10^{-3}$	$1.0685\text{x}10^{-3}$	$1.0152\text{x}10^{-3}$
$N = 256, k = 1$	$2.4283\text{x}10^{-3}$	$2.4074\text{x}10^{-3}$	$2.3959\text{x}10^{-3}$
$k = 3$	$5.6440\text{x}10^{-4}$	$5.4674\text{x}10^{-4}$	$5.3390\text{x}10^{-4}$

5. CONCLUSION

In our work, the exact closed-form frequency variance expressions of the PHD and k-PHD methods for a single real sinusoid in an additive white Gaussian noise for the random phase case have been derived .The phase of the sinusoid has been considered to be a random variable uniformly distributed in an interval of length 2π. Approximate simple formulas of the PHD and k-PHD variances of frequency for large data lengths and high signal-to-noise ratios have also been developed. The obtained variances of the frequency estimates are proved using computer simulations for various frequency values, data lengths and noise conditions.

REFERENCES

Anarim, E., and Istefanopulos, Y. 1991 Statistical analysis of pisarenko type tone frequency estimator, Signal Process., 24, 3: 291–298.

Chan, K. W. and So, H. C. 2003 An exact analysis of Pisarenko's single-tone frequency estimation algorithm, *signal processing, 83(3), 685-690. doi: 10.1016/S0165-1684(02)00493-0*

Chan, K., and So, H.-C. 2003 An exact analysis of pisarenko's single-tone frequency estimation algorithm, Signal Process., 83, 3: 685–690.

De Sabata, A., Toma, L, and Mischie, S. 2007 A new two-step single tone frequency estimation algorithm, international journal of Circuits, *Systems and Signal Processing, 2(1): 105-108.*

Keyta, B. M. and Dilaveroğlu, E. 2019 An exact variance expression of Pisarenko's method for frequency estimation of a single real sinusoid: Random Phase Case, to appear in the *Proceedings of the 3rd İnternational Symposium on İnnovative Approaches in Scientific Studies, Ankara, Turkey.*

Lui, K. W. K., and So, H. C. 2008 Modified pisarenko harmonic decomposition for single-tone frequency estimation," IEEE Trans. Signal Process., 56, 7: 3351–3356.

Makhoul, J. 1981 On the eigenvectors of symmetric toeplitz matrices, IEEE Trans. Acoust., Speech, Signal Process., 29, 4: 868–872.

Papoulis, A. 1991 Probability, Random Variables, and Stochastic Processes, *3rd ed., McGraw-Hill, New York.*

Sakai, H. 1984 Statistical analysis of Pisarenko's method for sinusoidal frequency estimation, *İEEE Transactions on Acoustics Speech, and Signal Processing, 32(1): 95-101.*

So, H. C., Chan, Y. T., Ho, K. C., and Chen, Y. 2013 Simple formulas for bias and mean square error computation, *İEEE Signal Processing Magazine, 30(4), 162-165. doi: 10.1109/MSP.2013.2254600.*

So, H.-C. and Chan, K. W. 2004 Reformulation of pisarenko harmonic decomposition method for single tone frequency estimation, IEEE Trans. Signal Process., 52, 4: 1128–1135.

Toma, L., De Sabata, A. and Vsiu, G. 2007 Two-step procedure decreases variance in tone frequency estimation, *Revue Roumaine des Sciences Techniques, Série Électrotechnique et Énergétique, 52(4) : 463-474.*

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